

Radiation Reaction on Brownian Motions

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Abstract

Tracking the “real” trajectory of a quantum particle still has been treated as the interpretation problem. It shall be expressed by the Brownian (stochastic) motion suggested by E. Nelson, but the well-defined field generation mechanism from a stochastic particle hasn’t been proposed. For the improvement of this, I propose the extension of Nelson’s quantum dynamics for a relativistic spin-less electron with its radiation, which is equivalent to the Klein-Gordon particle and field system. The accomplishment of this method can describe the general scattering of a single electron, namely, “radiation reaction” acting on a quantum-Brownian electron, which becomes important in high-intensity field physics by PW-class lasers at present. I show the formulation of this radiation reaction on a Brownian spin-less electron and the convergence to the previous models implementing the high-intensity field correction.

Keyword:

[Physics] Stochastic quantum dynamics, radiation reaction, high-intensity field physics

[mathematics] Applications of stochastic analysis

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Notation and Conventions

Symbol	Description
c	Speed of light
\hbar	Planck's constant
m_0	Rest mas of an electron
e	Charge of an electron
\mathbb{V}_M^4	4-dimentional standard vector space for the metric affine space
g	Metric on \mathbb{V}_M^4 with its signature $g := \text{sign}(+1, -1, -1, -1)$
$\mathbb{A}^4(\mathbb{V}_M^4, g)$	4-dimentional metric affine space with respect to \mathbb{V}_M^4 and g
$\mathcal{B}(I)$	Borel σ -algebra of a topological space I .
$(\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu)$	Measurable Minkowski spacetime
$(\Omega, D(\mathcal{P}), \mathcal{P})$	Probability space with the probability measure \mathcal{P}
$\mathbb{E}[\hat{X}(\bullet)] := \int_{\Omega} d\mathcal{P}(\omega) \hat{X}(\omega)$	Expectation of $\hat{X}(\bullet) := \{\hat{X}(\omega) \omega \in \Omega\}$
$\mathbb{E}[\hat{X}(\bullet) \mathcal{C}]$	Conditional expectation of $\hat{X}(\bullet)$ on $\mathcal{C} \subset D(\mathcal{P})$.
$\mathcal{P}_{\tau} \subset D(\mathcal{P})$	Sub- σ -algebra in the increasing family "Past" = $\{\mathcal{P}_{\tau} \tau > -\infty\}$
$\mathcal{F}_{\tau} \subset D(\mathcal{P})$	Sub- σ -algebra in the decreasing family "Future" = $\{\mathcal{F}_{\tau} \tau < \infty\}$
$\hat{x}(\circ, \bullet) := \{\hat{x}(\tau, \omega) \tau \in \mathbb{R}, \omega \in \Omega\}$	Dual progressive measurable process
	as the $\mathcal{B}(\mathbb{R}) \times D(\mathcal{P}) / \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g))$ measurable map
	(Collection of a spin-less electron's trajectory)
$\mathcal{V} \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$	Complex velocity; $\mathcal{V}^{\alpha}(x) := i\lambda^2 \times \partial^{\alpha} \ln \phi(x) + \frac{e}{m_0} A^{\alpha}(x)$

1 Introduction

This paper proposes the stochastic method of a radiating spin-less electron interacting with arbitrary light fields, which is equivalent to the coupled system of the Klein-Gordon equation and the Maxwell equation. Finally, we can find the quantum dynamics which can draw the trajectory of a radiating scalar electron in the general fields. Its generalization has become important in the research projects of the high-intensity lasers [1], represented by “Extreme Light Infrastructure (ELI)” [2, 3, 4] for the last laser-plasma science [5]. Corresponding to the name of “high-energy particle physics”, let us call such a high-intensity laser science “high-intensity field physics” in this paper. One of the key processes in this regime is the generalized scattering between an electron, incoming laser fields and radiation from this electron, named “radiation reaction”. Let us define this radiation reaction in quantum field theory.

Definition 1 (Radiation reaction). Consider the full Fock space $F(\mathcal{H})$ induced by the complex Hilbert space \mathcal{H} . Let S and $D(S) \subset F(\mathcal{H})$ be the S -matrix acting on Fock vectors and the domain of S . For the two Fock vectors $|\Psi\rangle = |e\rangle \otimes |\text{photons}\rangle$ and $|\Psi'\rangle = |e'\rangle \otimes |\text{photons}'\rangle$ generated from the single (scalar or spinor) electron states $|e\rangle, |e'\rangle \in \mathcal{H} \subset D(S)$ and arbitrary Bosonic Fock vectors $|\text{photons}\rangle, |\text{photons}'\rangle \in D(S)$ as light, the scattering class of “radiation reaction” is defined by the following relation:

$$\langle \Psi' | S | \Psi \rangle \neq 0 \quad (1)$$

As the famous application of this **Definition 1**, the non-linear Compton scattering has been calculated via the Furry picture:

Example 2 (Furry picture). Consider the Dirac equation

$$[\gamma_\mu (i\hbar\partial^\mu + eA_{\text{ex}}^\mu) - m_0 c \mathbb{I}^{4 \times 4}] \psi = 0 \quad (2)$$

and let the external field A_{ex} be “a plane wave”, the parameters \hbar, e, m_0, c be the physical constants and the 4×4 matrices $\gamma_{\mu=0,1,2,3}$ be Dirac’s gamma matrices. The solution of it is known well as the Volkov solution [6].

$$\psi_{\text{Volkov}}(x, p) = e^{-\frac{i}{\hbar} \mathcal{S}} \times \left[\mathbb{I}^{4 \times 4} - \frac{e(\gamma_\mu k^\mu) \cdot (\gamma_\nu A_{\text{ex}}^\nu)}{2p_\alpha k^\alpha} \right] \frac{u}{\sqrt{2p^0}} \quad (3)$$

$$\mathcal{S} = p_\mu x^\mu - \int^{\xi=k_\alpha x^\alpha} \frac{d\xi}{p_\alpha k^\alpha} \left(e p_\nu A_{\text{ex}}^\nu + \frac{e^2 g_{\mu\nu} A_{\text{ex}}^\mu A_{\text{ex}}^\nu}{2} \right) \quad (4)$$

Where u is a constant bi-spinor. Then, the set of the solutions $\{|\psi_{\text{Volkov}}(p)\rangle\} := \{\psi_{\text{Volkov}}(\circ, p)\}$ can construct the complex Hilbert space \mathcal{H} due to its orthogonality and completeness [8, 7]. Furthermore, $|\psi_{\text{Volkov}}(p)\rangle$ describes the absorption of the external field A_{ex} , $|\psi_{\text{Volkov}}(p)\rangle \in \mathcal{H}$ is employed instead of $|e\rangle \otimes |\text{photons}\rangle \in F(\mathcal{H})$. Let $\psi(x, p) := \psi_{\text{Volkov}}(x, p)|_{A_{\text{ex}}=0}$ be the Dirac spinor for the free propagation and S is the S -matrices acting on $\{|\psi(p)\rangle\}$. Then the modification $S \mapsto S^{\text{Furry}}$ is induced by the correspondence $\{|\psi(p)\rangle\} \mapsto \{|\psi_{\text{Volkov}}(p)\rangle\}$ like ref.[9], $|\psi_{\text{Volkov}}(p)\rangle$ can be applied to **Definition 1**. This scheme is so-called the Furry picture [9, 10]. As one of the examples, considering the lowest order of the radiation process

$$\langle \psi_{\text{Volkov}}(p') | \otimes \langle \text{photon}' | S_{\text{1st order}}^{\text{Furry}} | \psi_{\text{Volkov}}(p) \rangle \quad (5)$$

is named the non-linear Compton scattering [11, 12, 13] joining in the class $\langle \Psi' | S | \Psi \rangle \neq 0$. Where,

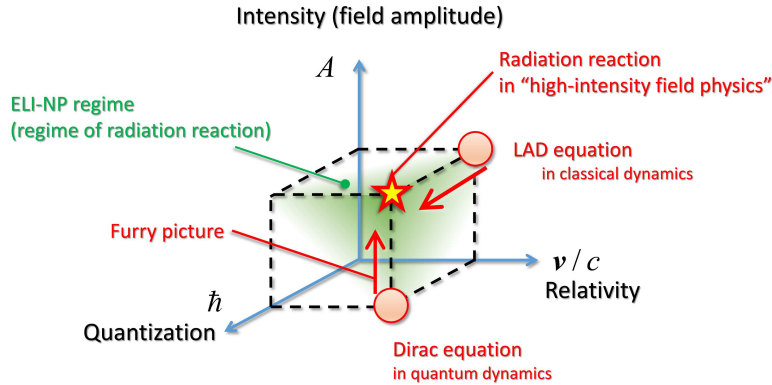


Figure 1: The physical regime of an electron. The point of “the star” is the regime of high-intensity field physics. The Furry picture (**Example 2**) is the first way to reach the goal. On the other hand as the second path, the quantization after reaching high-intensity “classical” dynamics is the candidate of it, too. The Lorentz-Abraham-Dirac (LAD) equation in high-intensity “classical” dynamics should be the standard model of radiation reaction. ELI-NP is the state-of-the-arts laser facility which has proposed the real experiments of radiation reaction (the green area in this figure) [2, 24]

■ $|\text{photon}'\rangle$ denotes the state of an emitted single photon.

By following the result of **Example 2**, many authors have investigated the models of radiation and a scattered electron [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26] based on this non-linear Compton scattering. Since the observations of them in high-intensity laser fields are quite important, the several new laser facilities have proposed to perform this real experiment (for example, see Ref.[24]).

However in the real computation, the orthogonality and the completeness of the Dirac-Volkov spinor interacting with “general photonic fields” is not obvious. It has not been known well whether can we construct its Hilbert space under such a generalized condition, yet. In the fact, there are the demonstrations only in the case of an external “plane wave” field [7, 8, 27]. Since strong focused and superpositioned lasers shall not be a plane wave in nature, hence the other new methods should be desired in high-intensity field physics.

Figure 1 is the strategy of the investigation of radiation reaction how to reach the regime of high-intensity field physics. The key parameters of high-intensity field physics are the speed of a particle v/c (the relativity), the Planck constant \hbar (the quantization) and the field amplitude A (the intensity). When all of these parameters act effectively on nature, high-intensity field physics appears. The Furry picture is the way from relativistic quantum dynamics (the Klein-Gordon equation/the Dirac equation). The second idea is the another path to reach high-intensity field physics via high-intensity “classical” electrodynamics. In this paper, let us stand this later perspective.

One of the naive ideas for realizing such a quantization from classical dynamics was provided by E. Nelson. His idea was that a quanta draws a Brownian motion as its trajectory in the non-relativistic regime [28, 29]. By employing $d\hat{\mathbf{x}}(t, \omega) = \mathbf{V}_{\pm}(\hat{\mathbf{x}}(t, \omega), t)dt + \sqrt{\hbar/2m_0} \times d\mathbf{W}_{\pm}(t, \omega)$, he succeeded to demonstrate not only (A)

his classical-style dynamics

$$m_0 \left[\partial_t \mathbf{v}(\mathbf{x}, t) + \mathbf{v}(\mathbf{x}, t) \cdot \nabla \mathbf{v}(\mathbf{x}, t) - \mathbf{u}(\mathbf{x}, t) \cdot \nabla \mathbf{u}(\mathbf{x}, t) - \frac{\hbar}{2m_0} \nabla^2 \mathbf{u}(\mathbf{x}, t) \right] = -\nabla \mathcal{V}(\mathbf{x}, t)$$

$$\mathbf{v}(\mathbf{x}, t) = \frac{\mathbf{V}_+(\mathbf{x}, t) + \mathbf{V}_-(\mathbf{x}, t)}{2}$$

$$\mathbf{u}(\mathbf{x}, t) = \frac{\mathbf{V}_+(\mathbf{x}, t) - \mathbf{V}_-(\mathbf{x}, t)}{2}$$

is equivalent to the Schrödinger equation, but also (B) he answered why the square of the wave function is regarded as the probability density [28, 29]. The biggest advantage of this method is (C) the ability of the tracking the “real” motion of a quantum particle. It also denotes (D) the transition between classical and quantum dynamics due to the semi-martingales. Since we have several classical models of radiation reaction, it seems to be well adopted by just borrowing the result from it. However, the feasibility of the coupled system between a stochastic particle and fields is not enough established. Hence, the realization of the field-generation from a stochastic charged particle must become a milestone for describing radiation reaction in high-intensity fields.

Let us recall radiation reaction as the coupling system in classical dynamics. The standard model of it was derived by P. A. M. Dirac, named the Lorentz-Abraham-Dirac (LAD) equation [30].

Theorem 3 (LAD equation). *Consider the single electron system in classical dynamics represented by the equation of motion and the Maxwell equation in the Minkowski spacetime:*

$$m_0 \frac{dv^\mu}{d\tau} = -e(F_{\text{ex}}^{\mu\nu} + F_{\text{LAD}}^{\mu\nu})v_\nu \quad (6)$$

$$\partial_\mu F^{\mu\nu} = \mu_0 \times \left[-ec \int_{\mathbb{R}} d\tau v^\nu(x) \delta^4(x - x(\tau)) \right] \quad (7)$$

Where m_0 , e , μ_0 and c are the physical constants. By solving the Maxwell equation, the retarded field F_{ret} and the advanced field F_{adv} are derived as its solutions:

$$\begin{aligned} F_{\text{ret}}(x(\tau)) &= \frac{e}{8\pi\epsilon_0 c^5} \left(\frac{dv}{d\tau} \otimes v - v \otimes \frac{dv}{d\tau} \right) \int_{\mathbb{R}} d\tau' \frac{\delta(\tau' - \tau)}{|\tau' - \tau|} \\ &\quad - \frac{e}{6\pi\epsilon_0 c^5} \left(\frac{d^2 v}{d\tau^2} \otimes v - v \otimes \frac{d^2 v}{d\tau^2} \right) \end{aligned} \quad (8)$$

$$\begin{aligned} F_{\text{adv}}(x(\tau)) &= \frac{e}{8\pi\epsilon_0 c^5} \left(\frac{dv}{d\tau} \otimes v - v \otimes \frac{dv}{d\tau} \right) \int_{\mathbb{R}} d\tau' \frac{\delta(\tau' - \tau)}{|\tau' - \tau|} \\ &\quad + \frac{e}{6\pi\epsilon_0 c^5} \left(\frac{d^2 v}{d\tau^2} \otimes v - v \otimes \frac{d^2 v}{d\tau^2} \right) \end{aligned} \quad (9)$$

By following the idea of Dirac, let F_{LAD} be the effective radiation reaction field characterized by the relation $F_{\text{LAD}} \equiv (F_{\text{ret}} - F_{\text{adv}})/2$, as the homogeneous solution of the Maxwell equation $\partial_\mu F_{\text{LAD}}^{\mu\nu} = 0$. Substituting F_{LAD} for the equation of motion, so-called, the Lorentz-Abraham-Dirac (LAD) equation [30] is induced as

the dynamics of a radiating spin-less (scalar) electron.

$$m_0 \frac{dv^\mu}{d\tau} = -eF_{\text{ex}}^{\mu\nu} v_\nu + \frac{m_0 \tau_0}{c^2} \left(\frac{d^2 v^\mu}{d\tau^2} v^\nu - \frac{d^2 v^\nu}{d\tau^2} v^\mu \right) v_\nu \quad (10)$$

Here, $\tau_0 = e^2/6\pi\epsilon_0 m_0 c^3 = O(10^{-24}\text{sec})$ in SI unit.

As the historical remark, this equation was developed due to the difficulties of the divergences in QED, namely, Dirac considered the modification of the standard classical equations before their quantization. Hence, our strategy of radiation reaction shall be said the revival of Dirac's considerations. The quantization from the LAD equation shall be the strong candidate beyond the Furry picture for radiation reaction. Due to this purpose, we discuss a possibility of the stochastic quantization of a radiating spin-less electron by following the meaning of Nelson. In order to the absence of the method how to treat the field-generation in it, the construction of it is also discussed.

In **Chapter 2**, we introduce the kinematics of a spin-less electron before exceeding its dynamics. At first, we define the D-progressively measurable process $\hat{x}(\circ, \bullet)$ as the trajectory class of the relativistic spin-less electron (the modification of Nelson's (S3)-process [29] for the Klein-Gordon equation), generated from the increasing and decreasing families of the σ -algebras. Then we can find the basic kinematics $d\hat{x}(\tau, \omega) = \mathcal{V}_\pm(\hat{x}(\tau, \omega))d\tau + \lambda \times dW_\pm(\tau, \omega)$. The complex velocity $\mathcal{V} := (1-i)/2 \times \mathcal{V}_+ + (1+i)/2 \times \mathcal{V}_-$ is introduced, and this new velocity \mathcal{V}^μ plays a role of the main cast in the present dynamics of a spin-less electron in **Chapter 3**. Since its trajectory is considered as the Brownian motion (the D-progressively measurable process), we must consider the Fokker-Planck equation for its diffusion process. In the end of this chapter, we discuss the most delicate problem how should we define the proper time holding the transition between classical physics to quantum physics.

For the calculation of $d\hat{x}(\circ, \bullet)$, we need to define the velocities $\mathcal{V}_\pm(\hat{x}(\circ, \bullet))$ and $\mathcal{V}(\hat{x}(\circ, \bullet))$. In **Chapter 3**, the dynamics of a quantum spin-less electron interacting with fields is the main scope. Hence, the mechanism of the field generation from a stochastic particle is also discussed here. Hence, our attention is dedicated to the purpose how to construct and demonstrate the Klein-Gordon equation and the Maxwell equation. Since we want to satisfy the transition between the classical and quantum regimes, the new functional (the action integral) are examined for that purpose. Hereby, the following dynamics of a spin-less particle and fields can be realized, as the transition from classical dynamics.

$$m_0 \mathfrak{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) = -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F^{\mu\nu}(\hat{x}(\tau, \omega))$$

$$\partial_\mu [F^{\mu\nu}(x) + \delta f^{\mu\nu}] = \mu_0 \times \mathbb{E} \left[-ec \int_{\mathbb{R}} d\tau \operatorname{Re} \{ \mathcal{V}^\nu(x) \} \delta^4(x - \hat{x}(\tau, \bullet)) \right]$$

Chapter 4 devotes to the derivation of the “radiation reaction” field acting on a stochastic spin-less electron as the scattered photons. It can be performed as the mimic of **Theorem 3** along the D-progressive $\hat{x}(\circ, \bullet)$. In order to this treatment, a quite similar dynamics of the LAD equation (10) can be found in the end. One of the key problem at here is, how to solve the Maxwell equation with the stochastic-valued index of the Green function. Let $\mathbb{E}[\hat{x}(\tau, \bullet)]$ be the expectation of $\hat{x}(\tau, \bullet)$ and \mathfrak{F} be the homogeneous radiation reaction field included in the field F . By the separation $\hat{x}(\tau, \omega) = \mathbb{E}[\hat{x}(\tau, \bullet)] + \delta\hat{x}(\tau, \omega)$, I propose the summation of the field acting on the expectation of its trajectories $\mathfrak{F}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ and the difference $\delta\hat{x}^\alpha(\tau, \omega) \cdot \partial_\alpha \mathfrak{F}(\mathbb{E}[\hat{x}(\tau, \bullet)])$, the equation of a radiating spin-less electron is formulated as follows via the definition $F(\hat{x}(\tau, \omega)) := F_{\text{ex}}(\hat{x}(\tau, \omega)) +$

$$\mathfrak{F}(\hat{x}(\tau, \omega)) + \delta \hat{x}^\alpha(\tau, \omega) \cdot \partial_\alpha \mathfrak{F}(\hat{x}(\tau, \omega)) + O(\otimes^2 \delta \hat{x}(\tau, \omega)):$$

$$\begin{aligned} m_0 \mathfrak{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) &= -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F_{\text{ex}}^{\mu\nu}(\hat{x}(\tau, \omega)) \\ &\quad - e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) \left[\begin{aligned} &\mathfrak{F}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \\ &+ \delta \hat{x}^\alpha(\tau, \omega) \cdot \partial_\alpha \mathfrak{F}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \end{aligned} \right] + O\left(\otimes^2 \delta \hat{x}(\tau, \omega)\right) \end{aligned}$$

For extracting its meaning in high-intensity field physics, the conclusion and discussion are prepared as **Chapter 5**. We can find that the new equation describes the field-dependence like [16, 17] by taking the expectation of dynamics (Ehrenfest's theorem) which is the most attractive result in high-intensity field physics. Furthermore in the realistic design of the experiments, it is expected some difficulties due to the higher order derivatives $d^n/d\tau^n \mathbb{E}[\hat{x}(\tau, \bullet)]$, $n = 2, 3, 4, \dots$ in the term of the radiation reaction force $-e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) \left[\mathfrak{F}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + \delta \hat{x}^\alpha(\tau, \omega) \cdot \partial_\alpha \mathfrak{F}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \right]$. Therefore, the perturbation of it is also proposed, corresponding to the method in classical dynamics, too.

2 Kinematics of a spin-less electron

The first part is the kinematics of a spin-less electron. Let $\mathbb{A}^4(\mathbb{V}_M^4, g)$ be the 4-dimensional metric affine space with respect to the 4-dimensional standard vector space \mathbb{V}_M^4 and the Minkowski metric g [31]. Defining the measurable space $(\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu)$, we consider this as the measurable Minkowski spacetime. Here $\mathcal{B}(I)$ denotes the Borel σ -algebra of a topological space I . For describing the quantum stochasticity of the trajectory, the probability space $(\Omega, D(\mathcal{P}), \mathcal{P})$ has to be defined, too [28, 29].

In these measurable spaces $(\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu)$ and $(\Omega, D(\mathcal{P}), \mathcal{P})$, the stochastic process of a spin-less electron $\hat{x}(\circ, \bullet)$, the complex velocity $\mathcal{V}(\hat{x}(\circ, \bullet))$, the Fokker-Planck equations and the proper time τ are discussed in this chapter. Let us regard the following descriptions that it has been introduced the coordinate map like $\varphi(x) := (x^0, x^1, x^2, x^3)$ even if we do not declare explicitly. And let $\mathbb{E}[\hat{X}(\bullet)]$ be the expectation of the stochastic process $\hat{X}(\bullet) := \{\hat{X}(\omega) | \omega \in \Omega\}$, $\mathbb{E}[\hat{X}(\bullet)] := \int_\Omega d\mathcal{P}(\omega) \hat{X}(\omega)$. $\mathbb{E}[\hat{X}(\bullet) | \mathcal{C}]$ is defined as the conditional expectation of $\hat{X}(\bullet)$ on $\mathcal{C} \subset D(\mathcal{P})$.

2.1 Stochastic process

Consider the stochastic process $\hat{x}(\circ, \bullet) := \{\hat{x}(\tau, \omega) | \tau \in \mathbb{R}, \omega \in \Omega\}$ as the map $\hat{x} : \mathbb{R} \times \Omega \rightarrow \mathbb{A}^4(\mathbb{V}_M^4, g)$. By following Ref.[28, 29, 32, 33, 34], let us introduce the sub- σ -algebras $\mathcal{P}_\tau, \mathcal{F}_\tau \subset D(\mathcal{P})$ and their filtration, the increasing family "Past" = $\{\mathcal{P}_\tau | \tau > -\infty\}$ and the decreasing family "Future" = $\{\mathcal{F}_\tau | \tau < \infty\}$. The stochastic process $\hat{x}(\circ, \bullet)$ can be regarded as the $\{\mathcal{P}_\tau\}$ -progressively measurable and the $\{\mathcal{F}_\tau\}$ -progressively measurable process.

Definition 4 (D-progressive $\hat{x}(\circ, \bullet)$). Consider the $\{\mathcal{P}_\tau\}$ -progressively measurable and the $\{\mathcal{F}_\tau\}$ -progressively measurable process $\hat{x}(\circ, \bullet)$.

[Nelson's (S1)]. For each $(\tau, \omega) \in \mathbb{R} \times \Omega$, when the following $\mathcal{B}((-\infty, \tau]) \times \mathcal{P}_\tau$ measurable function $\mathcal{V}_+^\mu(\hat{x}(\circ, \bullet))$ and the $\mathcal{B}([\tau, \infty)) \times \mathcal{F}_\tau$ measurable function $\mathcal{V}_-^\mu(\hat{x}(\circ, \bullet))$ exist as the limit in L^1 , $\hat{x}(\circ, \bullet)$ is named "Nelson's (S1)-process" [29]:

$$\mathcal{V}_+^\mu(\hat{x}(\tau, \omega)) = \lim_{\delta t \rightarrow 0+} \mathbb{E} \left[\left. \frac{\hat{x}^\mu(\tau + \delta t, \bullet) - \hat{x}^\mu(\tau, \bullet)}{\delta t} \right| \mathcal{P}_\tau \right] (\omega) \quad (11)$$

$$\mathcal{V}_-^\mu(\hat{x}(\tau, \omega)) = \lim_{\delta t \rightarrow 0+} \mathbb{E} \left[\left| \frac{\hat{x}^\mu(\tau, \bullet) - \hat{x}^\mu(\tau - \delta\tau, \bullet)}{\delta\tau} \right| \mathcal{F}_\tau \right] (\omega) \quad (12)$$

[D-progressive]. Let $W_+(\circ, \bullet)$ and $W_-(\circ, \bullet)$ be the forward and backward standard Wiener processes. For a given set $(\tau, \omega) \in \mathbb{R} \times \Omega$ and $\tau_a \leq \tau \leq \tau_b$, consider the following style like the Itô process [35].

$$\hat{x}^\mu(\tau, \omega) = \hat{x}^\mu(\tau_a, \omega) + \int_{\tau_a}^{\tau} d\tau' \mathcal{V}_+^\mu(\hat{x}(\tau', \omega)) + \lambda \times \int_{\tau_a}^{\tau} dW_+^\mu(\tau', \omega) \quad (13)$$

$$= \hat{x}^\mu(\tau_b, \omega) - \int_{\tau}^{\tau_b} d\tau' \mathcal{V}_-^\mu(\hat{x}(\tau', \omega)) - \lambda \times \int_{\tau}^{\tau_b} dW_-^\mu(\tau', \omega) \quad (14)$$

Where, $\lambda := \sqrt{\hbar/m_0} \in \mathbb{R}$ [36]. This stochastic process includes Nelson's (S1)-process obviously. Here, we introduce the modified rule of Nelson's (S2) and (S3)-processes [29] as the limit in L^2 :

$$g = - \lim_{\delta t \rightarrow 0+} \mathbb{E} \left[\left| \frac{[W_+(\tau + \delta\tau, \bullet) - W_+(\tau, \bullet)] \otimes [W_+(\tau + \delta\tau, \bullet) - W_+^\nu(\tau, \bullet)]}{\delta\tau} \right| \mathcal{P}_\tau \right] (\omega) \quad (15)$$

$$g = + \lim_{\delta t \rightarrow 0+} \mathbb{E} \left[\left| \frac{[W_-(\tau, \bullet) - W_-(\tau - \delta\tau, \bullet)] \otimes [W_-(\tau, \bullet) - W_-(\tau - \delta\tau, \bullet)]}{\delta\tau} \right| \mathcal{F}_\tau \right] (\omega) \quad (16)$$

We name “the dual-progressively measurable process”, or shortening “D-progressive” and also “the D-process”, such a $\{\mathcal{P}_\tau\}$ -progressive and $\{\mathcal{F}_\tau\}$ -progressive $\hat{x}(\circ, \bullet)$ (13-14) instead of Nelson's (S2) and (S3)-process [29]. Of cause, $g \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ is the metric in the Minkowski spacetime $(\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu)$ with its signature $g = \text{diag}(+1, -1, -1, -1)$. We also employ the differential form of (13-14):

$$\boxed{d\hat{x}^\mu(\tau, \omega) = \mathcal{V}_\pm^\mu(\hat{x}(\tau, \omega))d\tau + \lambda \times dW_\pm^\mu(\tau, \omega)} \quad (17)$$

By using this definition of the dual-progressively measurable process, let us assume the following idea of the relativistic kinematics for describing a spin-less electron.

Conjecture 5. *The D-progressive $\hat{x}(\circ, \bullet)$ draws the trajectory of a spin-less electron as the solution of the Klein-Gordon equation.*

The examination of **Conjecture 5** is the first topic what we need to demonstrate in this paper and we can find its feasibility in **Chapter 3**. Under the construction of the D-progressive $\hat{x}(\circ, \bullet)$, the standard Wiener processes $W_+(\circ, \bullet)$ and $W_-(\circ, \bullet)$ satisfy the following conditional expectations:

$$\begin{cases} \mathbb{E}[dW_+^\mu(\tau, \bullet) | \mathcal{P}_\tau] = 0 \\ \mathbb{E}[dW_-^\mu(\tau, \bullet) | \mathcal{F}_\tau] = 0 \end{cases} \quad (18)$$

These conditional expectations induce the expectation formula $\mathbb{E}[dW_\pm^\mu(\tau, \bullet)] = 0$, too. These standard Wiener processes in the D-process impose the following Itô rules [37];

$$d\tau \cdot d\tau = 0, \quad (19)$$

$$d\tau \cdot dW_\pm^\mu(\tau, \omega) = 0, \quad (20)$$

$$dW_\pm^\mu(\tau, \omega) \cdot dW_\pm^\nu(\tau, \omega) = \mp g^{\mu\nu} d\tau. \quad (21)$$

Moreover, let $\hat{\xi}_{\pm}(\tau, \omega)$ be the white noise as the time derivatives of $W_{\pm}(\tau, \omega)$ under meaning of the generalized-function satisfying $\int_{\mathbb{R}} d\tau d\Phi/d\tau(\tau, \omega) \cdot W_{\pm}^{\mu}(\tau, \omega) = -\int_{\mathbb{R}} d\tau \Phi(\tau, \omega) \cdot \hat{\xi}_{\pm}^{\mu}(\tau, \omega)$ with respect to a test function $\Phi(\tau, \omega)$ with respect to τ . By introducing the new symbols $d_{\pm}\hat{x}(\tau, \omega)$ as the RHS in (17), (17) can be recognized as the summation of the drift velocity $\mathcal{V}_{\pm}(\hat{x}(\tau, \omega))$ and the randomness $\lambda \times \hat{\xi}_{\pm}^{\mu}(\tau, \omega) = \lambda \times dW_{\pm}^{\mu}/d\tau(\tau, \omega)$,

$$\frac{d_{\pm}\hat{x}^{\mu}}{d\tau}(\tau, \omega) = \mathcal{V}_{\pm}^{\mu}(\hat{x}(\tau, \omega)) + \lambda \times \hat{\xi}_{\pm}^{\mu}(\tau, \omega). \quad (22)$$

Corresponding to (19-21), the modified Itô rule for this white noise can be derived as

$$\hat{\xi}_{\pm}^{\mu}(\tau, \omega) \hat{\xi}_{\pm}^{\nu}(\tau', \omega) = \mp g^{\mu\nu} \delta(\tau - \tau'). \quad (23)$$

Since $\mathbb{E}[\hat{\xi}_{+}^{\mu}(\tau, \bullet) | \mathcal{P}_{\tau}] = 0$ and $\mathbb{E}[\hat{\xi}_{-}^{\mu}(\tau, \bullet) | \mathcal{F}_{\tau}] = 0$, we repeatedly emphasize the conditional expectation of (22) at time τ (the mean-derivative) implies the drift velocities $\mathcal{V}_{\pm} \in \mathbb{V}_{\mathbb{M}}^4$:

$$\begin{aligned} \mathcal{V}_{+}^{\mu}(\hat{x}(\tau, \omega)) &:= \mathbb{E} \left[\left[\frac{d_{+}\hat{x}^{\mu}}{d\tau}(\tau, \bullet) \right] \middle| \mathcal{P}_{\tau} \right] (\omega) \\ &= \lim_{\delta t \rightarrow 0+} \mathbb{E} \left[\left[\frac{\hat{x}^{\mu}(\tau + \delta\tau, \bullet) - \hat{x}^{\mu}(\tau, \bullet)}{\delta\tau} \right] \middle| \mathcal{P}_{\tau} \right] (\omega) \end{aligned} \quad (24)$$

$$\begin{aligned} \mathcal{V}_{-}^{\mu}(\hat{x}(\tau, \omega)) &= \mathbb{E} \left[\left[\frac{d_{-}\hat{x}^{\mu}}{d\tau}(\tau, \bullet) \right] \middle| \mathcal{F}_{\tau} \right] \\ &:= \lim_{\delta t \rightarrow 0+} \mathbb{E} \left[\left[\frac{\hat{x}^{\mu}(\tau, \bullet) - \hat{x}^{\mu}(\tau - \delta\tau, \bullet)}{\delta\tau} \right] \middle| \mathcal{F}_{\tau} \right] (\omega) \end{aligned} \quad (25)$$

In general, the D-progressive $\hat{x}(\circ, \bullet)$ can induce the following Itô formula [31, 35].

Lemma 6 (Itô formula). *Consider the function $f : \mathbb{A}^4(\mathbb{V}_{\mathbb{M}}^4, g) \rightarrow \mathbb{C}$ as the C^2 -locally square integrable function $f \in L_{\text{loc}}^2(\mathbb{A}^4(\mathbb{V}_{\mathbb{M}}^4, g), \mu)$. Let $\partial_{\mu}f$ and $\partial_{\mu}\partial^{\mu}f$ also be the elements in $L_{\text{loc}}^2(\mathbb{A}^4(\mathbb{V}_{\mathbb{M}}^4, g), \mu)$ under the meaning of the generalized-function, $d_{\pm}f$ along the D-progressive $\hat{x}(\circ, \bullet)$ satisfies the following Itô formula with respect to (τ, ω) ;*

$$d_{\pm}f(\hat{x}(\tau, \omega)) = \partial_{\mu}f(\hat{x}(\tau, \omega))d_{\pm}\hat{x}^{\mu}(\tau, \omega) \mp \frac{\lambda^2}{2}\partial_{\mu}\partial^{\mu}f(\hat{x}(\tau, \omega))d\tau. \quad (26)$$

This is also expressed by the form of its integral,

$$f(\hat{x}(\tau_b, \omega)) - f(\hat{x}(\tau_a, \omega)) = \int_{\tau_a}^{\tau_b} d_{\pm}f(\hat{x}(\tau, \omega)) \quad (27)$$

$$= \int_{\tau_a}^{\tau_b} d_{\pm}\hat{x}^{\mu}(\tau, \omega) \partial_{\mu}f(\hat{x}(\tau, \omega)) \mp \frac{\lambda^2}{2} \int_{\tau_a}^{\tau_b} d\tau \partial_{\mu}\partial^{\mu}f(\hat{x}(\tau, \omega)). \quad (28)$$

2.2 Complex velocity

Especially by limiting the class in $\gamma_{\tau} = \mathcal{P}_{\tau} \cap \mathcal{F}_{\tau}$ as the “present” τ , the superposition of d_{+} and d_{-} is introduced. L. Nottale introduced that the following complex differential \hat{d} and the complex velocity $\mathcal{V}(\hat{x}(\circ, \bullet))$ as the essential manners in quantum dynamics [36].

Definition 7 (Complex differential and velocity). Consider the C^2 -locally square integrable function $f \in L_{\text{loc}}^2(\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g), \mu)$ such that $f : \mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g) \rightarrow \mathbb{C}$, the derivatives $d_+ f$ and $d_- f$ characterized by **Lemma 6**. Let \hat{d} be the complex differential defined at the point $\hat{x}(\tau, \omega)$ in $\gamma_\tau = \mathcal{P}_\tau \cap \mathcal{F}_\tau$ with the Markov property:

$$\hat{d} := \frac{1-i}{2}d_+ + \frac{1+i}{2}d_- \quad (29)$$

$$\hat{d}f(\hat{x}(\tau, \omega)) = \partial_\mu f(\hat{x}(\tau, \omega))\hat{d}\hat{x}^\mu(\tau, \omega) + \frac{i\lambda^2}{2}\partial^\mu \partial_\mu f(\hat{x}(\tau, \omega))d\tau \quad (30)$$

Then consider the conditional expectation of the derivative under the condition γ_τ is denoted by

$$\mathbb{E} \left[\left. \frac{\hat{d}f}{d\tau}(\hat{x}^\mu(\tau, \bullet)) \right| \gamma_\tau \right] (\omega) = \mathcal{V}^\mu(\hat{x}(\tau, \omega))\partial_\mu f(\hat{x}(\tau, \omega)) + \frac{i\lambda^2}{2}\partial^\mu \partial_\mu f(\hat{x}(\tau, \omega)), \quad (31)$$

especially when $f(\hat{x}(\tau, \omega)) = \hat{x}(\tau, \omega)$, it derives the complex velocity $\mathcal{V} \in \mathbb{V}_{\text{M}}^4 \oplus i\mathbb{V}_{\text{M}}^4$,

$$\mathcal{V}^\mu(\hat{x}(\tau, \omega)) := \mathbb{E} \left[\left. \frac{\hat{d}\hat{x}^\mu}{d\tau}(\tau, \bullet) \right| \gamma_\tau \right] (\omega) = \frac{1-i}{2}\mathcal{V}_+^\mu(\hat{x}(\tau, \omega)) + \frac{1+i}{2}\mathcal{V}_-^\mu(\hat{x}(\tau, \omega)). \quad (32)$$

Then choosing the wave function $\phi \in L_{\text{loc}}^2(\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g), \mu)$ like Ref.[36],

$$\mathcal{V}^\alpha(x) := i\lambda^2 \times \partial^\alpha \ln \phi(x) + \frac{e}{m_0}A^\alpha(x), \quad (33)$$

it connects to quantum dynamics.

This $\mathcal{V}^\alpha(x)$ behaves as the eigenvalue of the operator $i\hbar\mathcal{D}^\alpha/m_0 := [i\hbar\partial^\alpha + eA^\alpha(x)]/m_0$. It can be found the fact that ϕ satisfies the Klein-Gordon equation in the later study.

2.3 Fokker-Planck equations

Construct the probability space $(\Omega, D(\mathcal{P}), \mathcal{P})$ and the Minkowski spacetime $(\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)), \mu)$, then let us consider a C^2 -locally square integrable function $f \in L_{\text{loc}}^2(\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g), \mu)$ and its expectation $\mathbb{E}[\![f(\hat{x}(\circ, \bullet))]\!]$ on the D-progressive $\hat{x}(\circ, \bullet) : \mathbb{R} \times \Omega \rightarrow \mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)$. Where, the measure in the Minkowski spacetime is $\mu : \mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g) \rightarrow [0, \infty)$. From the definition of the expectation, a certain set $\Theta_\tau \subset \Omega$ and the $C^{2,1}$ -probability density function $p : \mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g) \times \mathbb{R} \rightarrow [0, \infty)$ characterized by the following relation should exist;

$$\mathcal{P}(\Theta_\tau) := \int_{\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)} d\mu(x) p(x, \tau) = 1. \quad (34)$$

Here, let Θ_τ be the set satisfying $\hat{x}(\Theta_\tau, \tau) \equiv \text{supp}(p(\circ, \tau)) \subset \mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)$. Since $\mathcal{P}(\Omega \setminus \Theta_\tau) \equiv 0$, we can expand (34) like $\mathcal{P}(\Omega) = \mathcal{P}(\Theta_\tau) = 1$. Hence, the expectation of $f(\hat{x}(\tau, \omega))$ is,

$$\mathbb{E}[\![f(\hat{x}(\tau, \bullet))]\!] := \int_\Omega d\mathcal{P}(\omega) f(\hat{x}(\tau, \omega)) = \int_{\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)} d\mu(x) f(x) p(x, \tau). \quad (35)$$

Consider the derivative of it by τ ,

$$\frac{d}{d\tau} \mathbb{E}[\![f(\hat{x}(\tau, \bullet))]\!] = \int_{\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)} d\mu(x) f(x) \partial_\tau p(x, \tau). \quad (36)$$

In the LHS of this equation (36), the evolution by $d_\pm \hat{x}(\tau, \omega)$ should be considered as follows;

$$\begin{aligned} \frac{d}{d\tau} \mathbb{E}[f(\hat{x}(\tau, \bullet))] &= \mathbb{E} \left[\mathcal{V}_\pm^\mu(\hat{x}(\tau, \bullet)) \partial_\mu f(\hat{x}(\tau, \bullet)) \mp \frac{\lambda^2}{2} \partial^\mu \partial_\mu f(\hat{x}(\tau, \bullet)) \right] \\ &= \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mu(x) f(x) \left\{ -\partial_\mu [\mathcal{V}_\pm^\mu(x) p(x, \tau)] \mp \frac{\lambda^2}{2} \partial^\mu \partial_\mu p(x, \tau) \right\}. \end{aligned} \quad (37)$$

For an arbitrary C^2 -function $f \in L_{\text{loc}}^2(\mathbb{A}^4(\mathbb{V}_M^4, g), \mu)$, the following Fokker-Planck equations along the D-progressive $\hat{x}(\circ, \bullet)$ are derived from (36) and (37).

Theorem 8 (Fokker-Planck equation). *Consider the D-progressive $\hat{x}(\circ, \bullet)$ in the probability space $(\Omega, D(\mathcal{P}), \mathcal{P})$, where the probability measure \mathcal{P} is characterized by (34). Let the $C^{2,1}$ -function $p : \mathbb{A}^4(\mathbb{V}_M^4, g) \times \mathbb{R} \rightarrow [0, \infty)$ be the probability density satisfying the following Fokker-Planck equation with respect to $x \in \mathbb{A}^4(\mathbb{V}_M^4, g)$:*

$$\partial_\tau p(x, \tau) + \partial_\mu [\mathcal{V}_\pm^\mu(x) p(x, \tau)] \pm \frac{\lambda^2}{2} \partial^\mu \partial_\mu p(x, \tau) = 0 \quad (38)$$

By using the definition of the complex velocity $\mathcal{V} \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$ (see (32)), we can consider the superposition of the “ \pm ”-Fokker-Planck equations by using the real and imaginary part of \mathcal{V} :

$$\partial_\tau p(x, \tau) + \partial_\mu [\text{Re}\{\mathcal{V}^\mu(x)\} p(x, \tau)] = 0 \quad (39)$$

$$\text{Im}\{\mathcal{V}^\mu(x)\} = \frac{\lambda^2}{2} \times \partial^\mu \ln p(x, \tau) \quad (40)$$

$$= \frac{\lambda^2}{2} \times \partial^\mu \ln \int_{\mathbb{R}} d\tau p(x, \tau), \quad (41)$$

$x \in \text{supp}(p(\circ, \tau))$

Equation (39) represents the equation of continuity of the probability density $p(x, \tau)$, (40-41) are a mimic of the osmotic pressure formula. The reason why there are the two expressions of (40) and (41), is the following should be derived from (38);

$$\text{Im}\{\mathcal{V}^\mu(x)\} p(x, \tau) - \frac{\lambda^2}{2} \partial^\mu p(x, \tau) = 0. \quad (42)$$

However, since this equation doesn't depend on the parameter τ ,

$$\text{Im}\{\mathcal{V}^\mu(x)\} \int_{\mathbb{R}} d\tau p(x, \tau) - \frac{\lambda^2}{2} \partial^\mu \int_{\mathbb{R}} d\tau p(x, \tau) = 0 \quad (43)$$

is also satisfied and derives the (41).

2.4 Proper time

One of the delicate problem in this paper is the definition of the proper time on the stochastic trajectory in the Minkowski spacetime $(\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu)$. Since we want to consider the D-process of a

particle as the quantization from classical dynamics, the limit $\hbar \rightarrow 0$ should induce the classical definition of the proper time. Before entering the main discussion, let us summarize it in classical dynamics.

$$d\tau|_{\text{classical}} = \frac{1}{c} \times \sqrt{dx_\mu(\tau)dx^\mu(\tau)} \quad (44)$$

Here, the metric is selected as $g = \text{diag}(+1, -1, -1, -1)$. At first for quantum dynamics, consider the following equation;

$$\hat{d}^* \hat{x}_\mu(\tau, \omega) \hat{d} \hat{x}^\nu(\tau, \omega) = \mathcal{V}_\mu^*(\hat{x}(\tau, \omega)) \mathcal{V}^\mu(\hat{x}(\tau, \omega)) d\tau^2. \quad (45)$$

Again, we recall the complex velocity

$$\mathcal{V}^\mu(x) = \frac{1}{m_0} \times \frac{i\hbar \partial^\mu \phi(x) + eA^\mu(x)\phi(x)}{\phi(x)} = \frac{1}{m_0} \times \frac{i\hbar \mathfrak{D}^\mu \phi(x)}{\phi(x)}. \quad (46)$$

Therefore, $\mathcal{V}_\mu^*(x)\mathcal{V}^\mu(x)$ becomes

$$\begin{aligned} \mathcal{V}_\mu^*(x)\mathcal{V}^\mu(x) &= \frac{1}{2m_0^2} \times \frac{\phi(x)(-i\hbar \mathfrak{D}_\mu^*) \cdot (-i\hbar \mathfrak{D}^{*\mu})\phi^*(x) + \phi^*(x)(i\hbar \mathfrak{D}_\mu) \cdot (i\hbar \mathfrak{D}^\mu)\phi(x)}{\phi^*(x)\phi(x)} \\ &\quad + \frac{\hbar^2}{2m_0^2} \times \frac{\partial_\mu \partial^\mu [\phi(x) \cdot \phi^*(x)]}{\phi^*(x)\phi(x)}. \end{aligned} \quad (47)$$

Let $\phi(x) \in L_{\text{loc}}^2(\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g), \mu)$ be the wave function of the complex Klein-Gordon equation, $(i\hbar \mathfrak{D}_\mu) \cdot (i\hbar \mathfrak{D}^\mu)\phi(x) - m_0^2 c^2 \phi(x) = 0$. The validity of this equivalency is given in **Ch. 3**. Due to this assumption, the first term of RHS in (47) must be a constant of c^2 . Then the issue is the behavior of $\hbar^2/m_0^2 \times \partial_\mu \partial^\mu [\phi^*(x)\phi(x)]/\phi^*(x)\phi(x)$. Where, we follow the proposal by T. Zastawniak in Ref.[38]. By defining the function $\phi(x) := \exp[R(x)/\hbar + iS(x)/\hbar]$, we obtain the relation $\phi^*(x)\phi(x) = \exp[2R(x)/\hbar]$. From the definition of (46), $\partial^\mu R(x) = \text{Im}\{m_0 \mathcal{V}^\mu(x)\} = \hbar/2 \times \partial^\mu \ln p(x, \tau)$ should be fulfilled (see (40)),

$$\frac{\hbar^2}{2m_0^2} \times \frac{\partial_\mu \partial^\mu [\phi(x) \cdot \phi^*(x)]}{\phi^*(x)\phi(x)} = \frac{\hbar^2}{2m_0^2} \times \frac{\partial_\mu \partial^\mu p(x, \tau)}{p(x, \tau)}, \quad (48)$$

hence, it is non-zero in general. However, let us introduce the expectation of (48) after the substitution $x = \hat{x}(\tau, \omega)$,

$$\begin{aligned} \mathbb{E} \left[\left[\frac{\hbar^2}{2m_0^2} \times \frac{\partial_\mu \partial^\mu p(\hat{x}(\tau, \bullet), \tau)}{p(\hat{x}(\tau, \bullet), \tau)} \right] \right] &= \frac{\hbar^2}{2m_0^2} \times \int_{\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)} d\mu(x) \left[\frac{\partial_\mu \partial^\mu p(x, \tau)}{p(x, \tau)} \right] p(x, \tau) \\ &= \frac{\hbar^2}{2m_0^2} \times \int_{\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)} d\mu(x) \partial_\mu \partial^\mu p(x, \tau) = 0, \end{aligned} \quad (49)$$

under the acceptable condition of $\partial^\mu p(x, \tau)|_{x \in \partial \mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)} = 0$. Here, $\partial \mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)$ denotes the boundary of $\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)$. Therefore the following relation is realized.

Lemma 9 (Lorentz invariant). *Consider the D-progressive $\hat{x}(\circ, \bullet)$. For all $\tau \in \mathbb{R}$, a relativistic-stochastic scalar electron satisfies the following invariant [38].*

$$\boxed{\mathbb{E} [\mathcal{V}_\mu^*(\hat{x}(\tau, \bullet)) \mathcal{V}^\mu(\hat{x}(\tau, \bullet))] = c^2} \quad (50)$$

This is the relation what we need to use instead of the classical relation (44). Due to this **Lemma 9**, the proper time is defined as the mimic of classical dynamics.

Definition 10 (Proper time). For all $\tau \in \mathbb{R}$, the proper time of a stochastic scalar electron is the following invariant parameter induced by **Lemma 9**;

$$d\tau := \frac{1}{c} \times \sqrt{\mathbb{E}[\hat{d}^* \hat{x}_\mu(\tau, \bullet) \cdot \hat{d} \hat{x}^\mu(\tau, \bullet)]}. \quad (51)$$

Here I want to mention the fact that the **Definition 10** realizes the transition between classical dynamics and quantum dynamics.

3 Dynamics of a spin-less electron and fields

In order to realize this kinematics (17), $d\hat{x}(\tau, \omega) = \mathcal{V}_\pm(\hat{x}(\tau, \omega))d\tau + \lambda \times dW_\pm(\tau, \omega)$, we need to investigate the behavior of the complex vector $\mathcal{V}^\mu(\hat{x}(\tau, \omega)) \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$. For the derivation of $\mathcal{V}(\hat{x}(\tau, \omega))$, the action integral (the functional) on a stochastic trajectory shall be required. Before entering the main discussion, we consider the variation of the action integral. After this explanation, let us proceed the concrete style of the action integral corresponding to the styles in classical dynamics.

3.1 Euler-Lagrange (Yasue) equation

In this small section, we focus the action integral excluding the field propagation. Hence, our present interest is a particle interacting with a field. For the method with the complex velocity $\mathcal{V} \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$, L. Nottale suggests the following style of the Lagrangian; $L_0(\tau, \hat{x}, \mathcal{V}_+, \mathcal{V}_-) = L(\tau, \hat{x}, \mathcal{V})$ due to the forward and the backward evolution. However, I propose the extension of it, $L_0(\tau, \hat{x}, \mathcal{V}_+, \mathcal{V}_-) = L(\tau, \hat{x}, \mathcal{V}, \mathcal{V}^*)$. Here $\mathcal{V}^* \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$ is the complex conjugate of \mathcal{V} . Recalling the definition of \mathcal{V} in (32),

$$L_0(\tau, \hat{x}, \mathcal{V}_+, \mathcal{V}_-) = L\left(\tau, \hat{x}, \frac{1-i}{2}\mathcal{V}_+ + \frac{1+i}{2}\mathcal{V}_-, \frac{1+i}{2}\mathcal{V}_+ + \frac{1-i}{2}\mathcal{V}_-\right) \quad (52)$$

along a sample path in the D-progressive $\hat{x}(\circ, \bullet)$. For the simplification, the following signatures are introduced ($\gamma_\tau = \mathcal{P}_\tau \cap \mathcal{F}_\tau$ as the “present” τ).

$$\mathfrak{D}_\tau^\pm := \mathbb{E}\left[\left[\frac{d_\pm}{d\tau}\right] \middle| \gamma_\tau\right] \quad (53)$$

$$\mathfrak{D}_\tau := \mathbb{E}\left[\left[\frac{\hat{d}}{d\tau}\right] \middle| \gamma_\tau\right] = \frac{1-i}{2}\mathfrak{D}_\tau^+ + \frac{1+i}{2}\mathfrak{D}_\tau^- \quad (54)$$

By using these expressions, $\mathfrak{D}_\tau^\pm \hat{x}^\mu(\tau, \omega) = \mathcal{V}_\pm^\mu(\hat{x}(\tau, \omega))$ is obviously satisfied. The variation of the functional $\int_{\tau_1}^{\tau_2} d\tau \mathbb{E}[L_0(\tau, \hat{x}, \mathcal{V}_+, \mathcal{V}_-)]$ with respect to \hat{x} is

$$\delta \int_{\tau_1}^{\tau_2} d\tau \mathbb{E}[L_0(\tau, \hat{x}, \mathcal{V}_+, \mathcal{V}_-)] = \int_{\tau_1}^{\tau_2} d\tau \mathbb{E}\left[\left[\frac{\partial L_0}{\partial \hat{x}^\mu} \delta \hat{x}^\mu + \frac{\partial L_0}{\partial \mathcal{V}_+^\mu} \delta \mathcal{V}_+^\mu + \frac{\partial L_0}{\partial \mathcal{V}_-^\mu} \delta \mathcal{V}_-^\mu\right]\right]$$

$$= \int_{\tau_1}^{\tau_2} d\tau \mathbb{E} \left[\left[\frac{\partial L}{\partial \hat{x}^\mu} \delta \hat{x}^\mu + \frac{\partial L}{\partial \mathcal{V}^\mu} \mathfrak{D}_\tau \delta \hat{x}^\mu + \frac{\partial L}{\partial \mathcal{V}^{*\mu}} \mathfrak{D}_\tau^* \delta \hat{x}^\mu \right] \right], \quad (55)$$

where, the following relations are introduced.

$$\frac{\partial L_0}{\partial \hat{x}^\mu} = \frac{\partial L}{\partial \hat{x}^\mu} \quad (56)$$

$$\frac{\partial L_0}{\partial \mathcal{V}_+^\mu} = \frac{1+i}{2} \frac{\partial L}{\partial \mathcal{V}^\mu} + \frac{1-i}{2} \frac{\partial L}{\partial \mathcal{V}^{*\mu}} \quad (57)$$

$$\frac{\partial L_0}{\partial \mathcal{V}_-^\mu} = \frac{1-i}{2} \frac{\partial L}{\partial \mathcal{V}^\mu} + \frac{1+i}{2} \frac{\partial L}{\partial \mathcal{V}^{*\mu}} \quad (58)$$

Then, we need to recall the following Nelson's partial integral [28, 29, 33].

Lemma 11 (Nelson's partial integral). *Let $\alpha, \beta \in L_{\text{loc}}^2(\mathbb{A}^4(\mathbb{V}_M^4, g), \mu)$ be the C^2 -local square integrable functions defined along the D -progressive $\hat{x}(\circ, \bullet)$, the following partial integral formula is fulfilled;*

$$\begin{aligned} \int_{\tau_1}^{\tau_2} d\tau \mathbb{E} \left[\left[\mathfrak{D}_\tau^\pm \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) + \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \mathfrak{D}_\tau^\mp \beta^\mu(\hat{x}(\tau, \bullet)) \right] \right] \\ = \mathbb{E} \left[\alpha_\mu(\hat{x}(\tau_2, \bullet)) \beta^\mu(\hat{x}(\tau_2, \bullet)) - \alpha_\mu(\hat{x}(\tau_1, \bullet)) \beta^\mu(\hat{x}(\tau_1, \bullet)) \right], \end{aligned} \quad (59)$$

or the form of its differential,

$$\frac{d}{d\tau} \mathbb{E} \left[\alpha_\mu(\hat{x}(\tau, \bullet)) \beta^\mu(\hat{x}(\tau, \bullet)) \right] = \mathbb{E} \left[\left[\mathfrak{D}_\tau^\pm \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) + \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \mathfrak{D}_\tau^\mp \beta^\mu(\hat{x}(\tau, \bullet)) \right] \right]. \quad (60)$$

By using the superposition of these “ \pm ”-formulas, it can be switched to the formula for the complex derivatives.

$$\begin{aligned} \frac{d}{d\tau} \mathbb{E} \left[\alpha_\mu(\hat{x}(\tau, \bullet)) \beta^\mu(\hat{x}(\tau, \bullet)) \right] &= \mathbb{E} \left[\left[\mathfrak{D}_\tau \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) + \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \mathfrak{D}_\tau^* \beta^\mu(\hat{x}(\tau, \bullet)) \right] \right] \\ &= \mathbb{E} \left[\left[\mathfrak{D}_\tau^* \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) + \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \mathfrak{D}_\tau \beta^\mu(\hat{x}(\tau, \bullet)) \right] \right] \end{aligned} \quad (61)$$

Proof. Consider the following relation at first;

$$\begin{aligned} \mathbb{E} \left[\left[\mathfrak{D}_\tau^+ \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) + \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \mathfrak{D}_\tau^- \beta^\mu(\hat{x}(\tau, \bullet)) \right] \right] \\ = \mathbb{E} \left[\left[\mathfrak{D}_\tau^- \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) + \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \mathfrak{D}_\tau^+ \beta^\mu(\hat{x}(\tau, \bullet)) \right] \right], \end{aligned} \quad (62)$$

since

$$\begin{aligned} \mathbb{E} \left[\left[\mathfrak{D}_\tau^+ \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) + \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \mathfrak{D}_\tau^- \beta^\mu(\hat{x}(\tau, \bullet)) \right] \right] \\ - \mathbb{E} \left[\left[\mathfrak{D}_\tau^- \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) + \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \mathfrak{D}_\tau^+ \beta^\mu(\hat{x}(\tau, \bullet)) \right] \right] \\ = -\lambda^4 \times \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mu(x) \partial^\nu \left\{ p(x, \tau) \begin{bmatrix} \partial_\nu \alpha_\mu(x) \cdot \beta^\mu(x) \\ -\alpha_\mu(x) \cdot \partial_\nu \beta^\mu(x) \end{bmatrix} \right\} = 0. \end{aligned} \quad (63)$$

Then using the Fokker-Planck equation (38),

$$\begin{aligned}
& \frac{d}{d\tau} \mathbb{E} [\alpha_\mu(\hat{x}(\tau, \bullet)) \beta^\mu(\hat{x}(\tau, \bullet))] \\
&= \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mu(x) \alpha_\mu(x) \beta^\mu(x) \partial_\tau p(x, \tau) \\
&= \frac{1}{2} \times \mathbb{E} \left[\left[(\mathfrak{D}_\tau^+ + \mathfrak{D}_\tau^-) \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot \beta^\mu(\hat{x}(\tau, \bullet)) \right] \right. \\
&\quad \left. + \alpha_\mu(\hat{x}(\tau, \bullet)) \cdot (\mathfrak{D}_\tau^+ + \mathfrak{D}_\tau^-) \beta^\mu(\hat{x}(\tau, \bullet)) \right] .
\end{aligned} \tag{64}$$

By combining it with (62), (60) is demonstrated. And also considering the superposition of (60), (61) is also imposed. \square

By considering (61), (55) becomes

$$\begin{aligned}
\delta \int_{\tau_1}^{\tau_2} d\tau \mathbb{E} [L_0(\tau, \hat{x}, \mathcal{V}_+, \mathcal{V}_-)] &= \int_{\tau_1}^{\tau_2} d\tau \mathbb{E} \left[\left(\frac{\partial L}{\partial \hat{x}^\mu} - \mathfrak{D}_\tau^* \frac{\partial L}{\partial \mathcal{V}^\mu} - \mathfrak{D}_\tau \frac{\partial L}{\partial \mathcal{V}^{*\mu}} \right) \delta \hat{x}^\mu \right] \\
&+ \int_{\tau_1}^{\tau_2} d\tau \frac{d}{d\tau} \mathbb{E} \left[\frac{\partial L}{\partial \mathcal{V}^\mu} \delta \hat{x}^\mu + \frac{\partial L}{\partial \mathcal{V}^{*\mu}} \delta \hat{x}^\mu \right] ,
\end{aligned} \tag{65}$$

due to the boundary conditions $\delta \hat{x}^\mu(\tau_{1,2}, \bullet) = 0$, the following **Theorem 12** should be derived.

Theorem 12 (Euler-Lagrange (Yasue) equation). *Let the functional*

$$\mathfrak{S}[\hat{x}, \mathcal{V}, \mathcal{V}^*] = \int_{\tau_1}^{\tau_2} d\tau \mathbb{E} [L(\tau, \hat{x}(\tau, \bullet), \mathcal{V}(\hat{x}(\tau, \bullet)), \mathcal{V}^*(\hat{x}(\tau, \bullet)))] \tag{66}$$

be the action integral of a particle along the D-progressive $\hat{x}(\circ, \bullet)$. By the variation of this action integral with respect to \hat{x} , the following Euler-Lagrange (Yasue) equation is induced:

$$\boxed{\frac{\partial L}{\partial \hat{x}^\mu} - \mathfrak{D}_\tau^* \frac{\partial L}{\partial \mathcal{V}^\mu} - \mathfrak{D}_\tau \frac{\partial L}{\partial \mathcal{V}^{*\mu}} = 0} \tag{67}$$

This is the version of the extended Euler-Lagrange equation for a stochastic particle, namely, this equation (67) corresponds to the Yasue equation in Nelson's framework [33, 34].

3.2 Action integral

In the Minkowski spacetime $(\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu)$, let us consider the action integral of “classical” dynamics;

$$\begin{aligned}
S_{\text{classical}} &= \int_{\mathbb{R}} d\tau \frac{m_0}{2} v_\alpha(\tau) v^\alpha(\tau) - \int_{\mathbb{R}} d\tau e A_\alpha(x(\tau)) v^\alpha(\tau) \\
&\quad + \frac{1}{4\mu_0 c} \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mu(x) F_{\alpha\beta}(x) F^{\alpha\beta}(x) .
\end{aligned} \tag{68}$$

Corresponding to (68), I propose the new action integral for Klein-Gordon particle-field system via the introduction of the mass measure and the charge measure.

Definition 13 (Mass and charge measures). Let \mathfrak{M} and \mathfrak{E} be the mass measure and the charge measure of a stochastic particle defined in the Minkowski spacetime $(\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu)$. For the positive constants m_0 and e with respect to $\tau \in \mathbb{R}$, \mathfrak{M} and \mathfrak{E} are characterized by

$$\int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mathfrak{M}(x, \tau) := m_0 \times \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mu(x) \mathbb{E} [\delta^4(x - \hat{x}(\tau, \bullet))], \quad (69)$$

$$\int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mathfrak{E}(x, \tau) := e \times \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mu(x) \mathbb{E} [\delta^4(x - \hat{x}(\tau, \bullet))]. \quad (70)$$

The following is also introduced for simply writing,

$$d\mathfrak{M}(x, \tau) = m_0 \times \mathbb{E} [\delta^4(x - \hat{x}(\tau, \bullet))] d\mu(x), \quad (71)$$

$$d\mathfrak{E}(x, \tau) = e \times \mathbb{E} [\delta^4(x - \hat{x}(\tau, \bullet))] d\mu(x). \quad (72)$$

The key of this definition is the appearance of the smoothed distribution $\mathbb{E} [\delta^4(x - \hat{x}(\tau, \bullet))] d\mu(x)$ from $\delta^4(x - x(\tau))d\mu(x)$ in classical dynamics. By this new idea with the complex velocity $\mathcal{V}(x) \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$ (32-33), let the following functional \mathfrak{S} be the action integral of the dynamics.

Theorem 14 (Lagrangian I). *The following functional \mathfrak{S} is the action integral deriving the dynamics between a “stochastic” spin-less electron and the field characterized by $\mathcal{V} \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$, $A \in \mathbb{V}_M^4$, $F \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ and the given tensor $\delta f \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$.*

$$\begin{aligned} \mathfrak{S}[\hat{x}, \mathcal{V}, \mathcal{V}^*, A] &= \frac{1}{2} \int_{\mathbb{R}} d\tau \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mathfrak{M}(x, \tau) \mathcal{V}_\alpha^*(x) \mathcal{V}^\alpha(x) \\ &\quad - \int_{\mathbb{R}} d\tau \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mathfrak{E}(x, \tau) A_\alpha(x) \text{Re} \{ \mathcal{V}^\alpha(x) \} \\ &\quad + \frac{1}{4\mu_0 c} \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mu(x) [F_{\alpha\beta}(x) + \delta f_{\alpha\beta}(x)] \cdot [F^{\alpha\beta}(x) + \delta f^{\alpha\beta}(x)] \end{aligned} \quad (73)$$

Writing the detail of the measures explicitly,

$$\begin{aligned} \mathfrak{S}[\hat{x}, \mathcal{V}, \mathcal{V}^*, A] &= \mathbb{E} \left[\int_{\mathbb{R}} d\tau \frac{m_0}{2} \mathcal{V}_\alpha^*(\hat{x}(\tau, \bullet)) \mathcal{V}^\alpha(\hat{x}(\tau, \bullet)) \right] \\ &\quad + \mathbb{E} \left[- \int_{\mathbb{R}} d\tau e A_\alpha(\hat{x}(\tau, \bullet)) \text{Re} \{ \mathcal{V}^\alpha(\hat{x}(\tau, \bullet)) \} \right] \\ &\quad + \frac{1}{4\mu_0 c} \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mu(x) [F_{\alpha\beta}(x) + \delta f_{\alpha\beta}(x)] \cdot [F^{\alpha\beta}(x) + \delta f^{\alpha\beta}(x)] \end{aligned} \quad (74)$$

Where, $F^{\alpha\beta}(x) := \partial^\alpha A^\beta - \partial^\beta A^\alpha$ and $\delta f^{\alpha\beta}(x) := \partial^\alpha \delta a^\beta - \partial^\beta \delta a^\alpha$ with respect to the arbitrary 4-vector $\delta a \in \mathbb{V}_M^4$.

3.3 Dynamics of a spin-less electron

Here, the action integral of the stochastic particle with the interaction is

$$\begin{aligned} \mathfrak{S}_{\text{particle}}[\hat{x}, \mathcal{V}, \mathcal{V}^*, A] &= \frac{1}{2} \int_{\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)} d\mathfrak{M}(x, \tau) \mathcal{V}_\alpha^*(x) \mathcal{V}^\alpha(x) \\ &\quad - \int_{\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)} d\mathfrak{E}(x, \tau) A_\alpha(x) \text{Re} \{ \mathcal{V}^\alpha(x) \} . \end{aligned} \quad (75)$$

Substituting this for (67), we can find the equation

$$\text{Re} \left\{ m_0 \mathfrak{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) + e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F^{\mu\nu}(\hat{x}(\tau, \omega)) \right\} = 0 . \quad (76)$$

Here being introduced the following new signature $\hat{\mathcal{V}}^\mu(x)$ and the relations, with the Lorenz gauge $\partial_\mu A^\mu = 0$ [36]:

$$\hat{\mathcal{V}}^\mu(x) := \mathcal{V}^\mu(x) + \frac{i\lambda^2}{2} \partial^\mu \quad (77)$$

$$\mathfrak{D}_\tau = \hat{\mathcal{V}}^\mu(x) \partial_\mu \quad (78)$$

$$\mathfrak{D}_\tau A_\mu(\hat{x}(\tau, \omega)) = \hat{\mathcal{V}}^\nu(\hat{x}(\tau, \omega)) \partial_\nu A_\mu(\hat{x}(\tau, \omega)) \quad (79)$$

$$\mathfrak{D}_\tau^* A_\mu(\hat{x}(\tau, \omega)) = \hat{\mathcal{V}}^{*\nu}(\hat{x}(\tau, \omega)) \partial_\nu A_\mu(\hat{x}(\tau, \omega)) \quad (80)$$

Theorem 15 (Equation of a stochastic particle's motion). *The equation of a “stochastic” motion of a spin-less electron interacting with fields is*

$$\boxed{d\mathfrak{M}(x, \tau) \mathfrak{D}_\tau \mathcal{V}^\mu(x) = -d\mathfrak{E}(x, \tau) \hat{\mathcal{V}}_\nu(x) F^{\mu\nu}(x)} \quad (81)$$

derived from the action integral (73-74). Considering the integral with respect to $x \in \mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)$,

$$\boxed{m_0 \mathfrak{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) = -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F^{\mu\nu}(\hat{x}(\tau, \omega))} . \quad (82)$$

These equations are equivalent to the Klein-Gordon equation.

Proof. Let an arbitrary smooth $C^{1,0}$ -function $f : \mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g) \times \mathbb{R} \rightarrow \mathbb{R}$ be a degree of freedom of the imaginary part in (76), namely,

$$m_0 \mathfrak{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) = -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F^{\mu\nu}(\hat{x}(\tau, \omega)) + \frac{i}{2m_0^2} \partial^\mu f(\hat{x}(\tau, \omega), \tau) . \quad (83)$$

This equation is the general solution of (76). Transforming $\mathfrak{D}_\tau \mathcal{V}^\mu + e/m_0 \times \hat{\mathcal{V}}_\nu F^{\mu\nu}$ with (78), we can get the brief style

$$\mathfrak{D}_\tau \mathcal{V}^\mu + \frac{e}{m_0} \hat{\mathcal{V}}_\nu F^{\mu\nu} = \hat{\mathcal{V}}_\nu \left[\partial^\nu \mathcal{V}^\mu + \frac{e}{m_0} F^{\mu\nu} \right] = \underline{\hat{\mathcal{V}}_\nu \partial^\mu \mathcal{V}^\nu} , \quad (84)$$

because of the identity

$$\partial^\alpha \mathcal{V}^\beta - \partial^\beta \mathcal{V}^\alpha = \frac{e}{m_0} F^{\alpha\beta} \quad (85)$$

derived from (33) (also see Ref[36]). Substituting (33) and (77) for this (83),

$$\begin{aligned}\hat{\mathcal{V}}_\nu \partial^\mu \mathcal{V}^\nu - \frac{i}{2m_0^2} \partial^\mu f &= \left[i\lambda^2 \times \partial_\nu \ln \phi + \frac{e}{m_0} A_\nu + \frac{i\lambda^2}{2} \partial_\nu \right] \times \partial^\mu \left[i\lambda^2 \times \partial^\nu \ln \phi + \frac{e}{m_0} A^\nu \right] - \frac{i}{2m_0^2} \partial^\mu f \\ &= \frac{1}{2} \partial^\mu \left[\frac{(i\hbar \partial_\nu + eA_\nu)(i\hbar \partial^\nu + eA^\nu) \phi - i f \phi}{m_0^2 \phi} \right] = 0.\end{aligned}\quad (86)$$

Therefore, we can find the quasi-Klein-Gordon equation by putting an arbitrary constant c^2 ,

$$(i\hbar \partial_\nu + eA_\nu)(i\hbar \partial^\nu + eA^\nu) \phi - (m_0^2 c^2 + i f) \phi = 0. \quad (87)$$

However, since the definition (33) supports **Lemma 9**

$$\mathbb{E} [\mathcal{V}_\mu^*(\hat{x}(\tau, \bullet)) \mathcal{V}^\mu(\hat{x}(\tau, \bullet))] = c^2, \quad (88)$$

and the Klein-Gordon equation

$$(i\hbar \partial_\nu + eA_\nu)(i\hbar \partial^\nu + eA^\nu) \phi - m_0^2 c^2 \phi = 0. \quad (89)$$

Due to this reason, the freedom of the imaginary part $i/2m_0^2 \times \partial^\mu f(\hat{x}(\tau))$ in (83) should be zero. Therefore the equation of motion (81 or 82) is equivalent to the Klein-Gordon equation. \square

This equation (81 or 82) is very close style of classical dynamics. Ehrenfest's theorem is implied as the average behavior of this stochastic spin-less electron.

Theorem 16 (Ehrenfest's theorem). *The expectation of the equation of motion (82) derives Ehrenfest's theorem for the Klein-Gordon equation.*

Proof. Due to the identity $\mathbb{E}[dW_\pm^\mu(\tau, \bullet)] = 0$, $\mathbb{E}[d\hat{x}^\mu(\tau, \bullet)] = \mathbb{E}[\mathcal{V}_\pm^\mu(\hat{x}(\tau, \bullet))]d\tau$ and $\mathbb{E}[\mathcal{V}_+^\mu(\hat{x}(\tau, \bullet))] = \mathbb{E}[\mathcal{V}_-^\mu(\hat{x}(\tau, \bullet))]$ are satisfied. Considering the expectation of the equation of motion (82),

$$\begin{aligned}m_0 \frac{d}{d\tau} \mathbb{E} [\mathcal{V}^\mu(\hat{x}(\tau, \bullet))] &= m_0 \frac{d}{d\tau} \mathbb{E} [\text{Re}\{\mathcal{V}^\mu(\hat{x}(\tau, \bullet))\}] \\ &\stackrel{\text{Eq.(61)}}{=} \text{Re}\{\mathbb{E}[m_0 \mathcal{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \bullet))]\} \\ &= \mathbb{E} [\text{Re}\{f_{\text{ex}}^\mu(\hat{x}(\tau, \bullet))\}].\end{aligned}\quad (90)$$

Where, $f_{\text{ex}}^\mu(\hat{x}(\tau, \omega)) := -e\hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega))F^{\mu\nu}(\hat{x}(\tau, \omega)) \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$. Since $d/d\tau \mathbb{E} [\hat{x}^\mu(\tau, \bullet)] = \mathbb{E} [\mathcal{V}^\mu(\hat{x}(\tau, \bullet))]$,

$$\boxed{m_0 \frac{d^2}{d\tau^2} \mathbb{E} [\hat{x}^\mu(\tau, \bullet)] = \mathbb{E} [\text{Re}\{f_{\text{ex}}^\mu(\hat{x}(\tau, \bullet))\}] \in \mathbb{V}_M^4} \quad (91)$$

should be satisfied and it is Ehrenfest's theorem which is the average motion of the Klein-Gordon particle. \square

Due to this **Theorem 16**, the correspondence of the velocities between classical and quantum dynamics is

$$\boxed{v^\mu(\tau) \leftrightarrow \frac{d}{d\tau} \mathbb{E} [\hat{x}(\tau, \bullet)] = \mathbb{E} [\text{Re}\{\mathcal{V}(\hat{x}(\tau, \bullet))\}]} \quad (92)$$

Furthermore in classical dynamics, $d/d\tau(v_\mu v^\mu) = 2 \times v_\mu dv^\mu/d\tau \equiv 0$ must be satisfied. The present dynamics of a stochastic particle provides a similar relation.

$$\begin{aligned}
\frac{d}{d\tau} \mathbb{E} [\mathcal{V}_\mu^*(\hat{x}(\tau, \bullet)) \mathcal{V}^\mu(\hat{x}(\tau, \bullet))] &= \mathbb{E} \left[\begin{aligned} &\mathcal{V}_\mu^*(\hat{x}(\tau, \bullet)) \cdot \mathfrak{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \bullet)) \\ &+ \mathfrak{D}_\tau^* \mathcal{V}_\mu^*(\hat{x}(\tau, \bullet)) \cdot \mathcal{V}^\mu(\hat{x}(\tau, \bullet)) \end{aligned} \right] \\
&= -\frac{\lambda^2 e}{m_0} \times \mathbb{E} \left[\text{Im} \{ \mathcal{V}_\mu(\hat{x}(\tau, \bullet)) \} \cdot \partial_\nu F^{\mu\nu}(\hat{x}(\tau, \bullet)) \right] \\
&= -\frac{\lambda^4 e}{2m_0} \times \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mu(x) \partial_\mu p(x, \tau) \cdot \partial_\nu F^{\mu\nu}(x) \\
&= \frac{\lambda^4 e}{2m_0} \times \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mu(x) p(x, \tau) \cdot \partial_\mu \partial_\nu F^{\mu\nu}(x) = 0
\end{aligned} \tag{93}$$

Where, the natural boundary condition $p(x, \tau)|_{x \in \partial \mathbb{A}^4(\mathbb{V}_M^4, g)} = 0$ is selected. This calculation also supports the Lorentz invariant $\mathbb{E} [\mathcal{V}_\mu^*(\hat{x}(\tau, \bullet)) \mathcal{V}^\mu(\hat{x}(\tau, \bullet))] = c^2$ (constant).

Lemma 17. *The trajectory of a stochastic spin-less electron satisfies the following relation;*

$$\boxed{\frac{d}{d\tau} \mathbb{E} [\mathcal{V}_\mu^*(\hat{x}(\tau, \bullet)) \mathcal{V}^\mu(\hat{x}(\tau, \bullet))] = 0}. \tag{94}$$

3.4 Dynamics of fields

Let us proceed the dynamics of the field interacting with a stochastic scalar electron. The Maxwell equation is derived by the variation of (74) with respect to $A \in \mathbb{V}_M^4$. The action integral for the field dynamics is,

$$\begin{aligned}
\mathfrak{S}_{\text{field}}[\hat{x}, A] &= - \int_{\mathbb{R}} d\tau \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mathfrak{E}(x, \tau) A_\alpha(x) \text{Re} \{ \mathcal{V}^\alpha(x) \} \\
&\quad + \frac{1}{4\mu_0 c} \int_{\mathbb{A}^4(\mathbb{V}_M^4, g)} d\mu(x) [F_{\alpha\beta}(x) + \delta f_{\alpha\beta}(x)] \cdot [F^{\alpha\beta}(x) + \delta f^{\alpha\beta}(x)].
\end{aligned} \tag{95}$$

Theorem 18 (Maxwell equation). *Let the D -progressive $\hat{x}(\circ, \bullet)$ be the trajectory of a stochastic spin-less electron. The variation of the action integral (74) with respect to the field $A \in \mathbb{V}_M^4$ derives the following Maxwell equation.*

$$\boxed{\partial_\mu [F^{\mu\nu}(x) + \delta f^{\mu\nu}(x)] = \mu_0 \times \mathbb{E} \left[-ec \int_{\mathbb{R}} d\tau \text{Re} \{ \mathcal{V}^\nu(x) \} \delta^4(x - \hat{x}(\tau, \bullet)) \right]} \tag{96}$$

Where, $\delta f \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ is the given field and the current of a stochastic spin-less electron

$$j_{\text{stochastic}}^\mu(x) := \mathbb{E} \left[-ec \int_{\mathbb{R}} d\tau \text{Re} \{ \mathcal{V}^\mu(x) \} \delta^4(x - \hat{x}(\tau, \bullet)) \right] \tag{97}$$

is equivalent to the current of Klein-Gordon particle

$$j_{\text{K-G}}^\mu(x) = -\frac{ie\lambda^2}{2} \times g^{\mu\nu} [\phi^*(x) \mathfrak{D}_\nu \phi(x) - \phi(x) \mathfrak{D}_\nu^* \phi^*(x)]. \tag{98}$$

Where, $\mathfrak{D}^\mu := \partial^\mu - ie/\hbar \times A^\mu(x)$.

Proof. The derivation of the Maxwell equation is obvious. The current $j_{\text{stochastic}}^\mu(x)$ is calculated as follows by employing (33):

$$\begin{aligned} j_{\text{stochastic}}^\mu(x) &= -ec \int_{\mathbb{R}} d\tau \operatorname{Re} \{ \mathcal{V}^\alpha(x) \} p(x, \tau) \\ &= \frac{\int_{\mathbb{R}} d\tau p(x, \tau)}{\phi^*(x)\phi(x)} \times j_{\text{K-G}}^\mu(x) \end{aligned} \quad (99)$$

Hereby, $j_{\text{stochastic}}(x) \in \mathbb{V}_{\text{M}}^4$ satisfies $\partial_\mu j_{\text{stochastic}}^\mu(x) = 0$ due to the equation of continuity (39) with the natural boundary condition $p(x, \tau = \partial\mathbb{R}) = 0$. Of cause, $\partial_\mu j_{\text{K-G}}^\mu(x) = 0$ should be held, too. Due to the divergences of these currents,

$$\frac{\int_{\mathbb{R}} d\tau p(x, \tau)}{\phi^*(x)\phi(x)} = \text{Constant} \quad (100)$$

should be fulfilled, the Maxwell equation with the current by a Klein-Gordon particle $\partial_\mu [F^{\mu\nu}(x) + \delta f^{\mu\nu}(x)] = \mu_0 j_{\text{K-G}}^\nu$ is realized by the selection $\int_{\mathbb{R}} d\tau p(x, \tau) / \phi^*(x)\phi(x) = 1$. \square

The following rule is implied by the above discussion.

Assumption 19. *For the realization of the Klein-Gordon equation and the Maxwell equation from the action integral (73-74), the following relation should be satisfied with respect to $x \in \operatorname{supp}(p(\circ, \tau))$:*

$$\begin{aligned} \phi^*(x)\phi(x) &:= \int_{\mathbb{R}} d\tau \mathbb{E} [\delta^4(x - \hat{x}(\tau, \bullet))] \\ &= \int_{\mathbb{R}} d\tau p(x, \tau) \end{aligned} \quad (101)$$

Here, the coupling system between a stochastic particle and fields can be regarded exactly as the system between a Klein-Gordon particle and fields.

4 Radiation reaction

The final topics in this paper is the “radiation reaction” effects along the D-progressive $\hat{x}(\circ, \bullet)$ as an application of this stochastic method. The standard model of radiation reaction is the LAD equation (**Theorem 3**) in classical dynamics, however its quantization has been getting important in the general high-intensity laser fields. From here, let us consider the derivation of the effective field strength tensor of radiation reaction as the mimic of classical dynamics like Ref.[30, 39]. What we need to get are the retarded field $\mathcal{F}_{(+)}(\hat{x}(\tau, \omega)) \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4$ and the advanced fields $\mathcal{F}_{(-)}(\hat{x}(\tau, \omega)) \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4$ as the solution of the Maxwell equation (96). Since the treatment of the stochastic valued index is complected, therefore we consider the expansion $\mathcal{F}_{(\pm)}(\hat{x}(\tau, \omega)) = \mathcal{F}_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + \delta \hat{x}^\alpha(\tau, \omega) \cdot \partial_\alpha \mathcal{F}_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + O(\otimes^2 \delta \hat{x}(\tau, \omega))$ around the average position $\mathbb{E}[\hat{x}(\tau, \bullet)]$ by introducing the difference $\delta \hat{x}(\tau, \omega) := \hat{x}(\tau, \omega) - \mathbb{E}[\hat{x}(\tau, \bullet)]$. Resulting this calculation, the wider scattering class than the Furry picture (**Example 2**) can be realized.

Considering the Green function as the solution of $\partial_\mu \partial^\mu G_{(\pm)}(x, x') = \delta^4(x - x')$, the solution of the Maxwell

equation $\partial_\mu \mathcal{F}_{(\pm)}^{\mu\nu}(x) = \mu_0 j_{\text{stochastic}}^\mu(x)$ (96) is

$$A_{(\pm)}(x) + \delta a_{(\pm)}(x) = -ec\mu_0 \int_{\mathbb{R}} d\tau' \mathbb{E} \left[\mathcal{V}_{\text{real}}(\hat{x}(\tau', \bullet)) G_{(\pm)}(x, \hat{x}(\tau', \bullet)) \right] \in \mathbb{V}_M^4 \quad (102)$$

under the Lorenz gauge $\partial_\nu A_{(\pm)}^\nu = 0$ and $\partial_\nu \delta a_{(\pm)}^\nu = 0$. Where, $G_{(\pm)}(x, x') = \theta(\pm \Delta x^0)/2\pi \times \delta(\Delta x_\mu \Delta x^\mu)$ and $\Delta x := x - x' \in \mathbb{V}_M^4$. The signature of “ $\dot{+}/\dot{-}$ ” represents the “retarded/advanced” Green function and $\mathcal{V}_{\text{real}} := \text{Re}\{\mathcal{V}\} \in \mathbb{V}_M^4$. By following the method of Ref.[39] and the relation $c^2 \varepsilon_0 \mu_0 = 1$ in electromagnetism, the field strength $\mathcal{F}_{(\pm)}^{\mu\nu}(x) := \partial^\mu [A_{(\pm)}^\nu(x) + \delta a_{(\pm)}^\nu(x)] - \partial^\nu [A_{(\pm)}^\mu(x) + \delta a_{(\pm)}^\mu(x)]$ becomes,

$$\mathcal{F}_{(\pm)}^{\mu\nu}(x) = -\frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \mathbb{E} \left[\left(\mathcal{V}_{\text{real}}^\nu(\hat{x}(\tau', \bullet)) \cdot \partial^\mu - \mathcal{V}_{\text{real}}^\mu(\hat{x}(\tau', \bullet)) \cdot \partial^\nu \right) G_{(\pm)}(x, \hat{x}(\tau', \bullet)) \right]. \quad (103)$$

Here, one of the question is the treatment of $\partial^\mu G_{(\pm)}(x, \hat{x}(\tau', \bullet))$. For solving this problem, let us consider the simple model, namely, $\int_{\mathbb{R}} d\tau' \mathbb{E}[f(\hat{x}(\tau', \bullet))]$ with respect to the $\mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g))/\mathcal{B}(\mathbb{R})$ -measurable function f before entering the main body of the discussion.

Theorem 20 (Smoothing factor). *In the measurable Minkowski spacetime $(\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu)$, consider the arbitrary C^∞ -local square integrable and $\mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g))/\mathcal{B}(\mathbb{R})$ -measurable function f along the D -progressive $\hat{x}(\circ, \bullet)$ in the meaning of the generalized-function. Then, a certain C^∞ -function $\Xi : \mathbb{R} \rightarrow \mathbb{R}$ exists such that*

$$\boxed{\mathbb{E}[f(\hat{x}(\tau, \bullet))] = \Xi(\tau) \times f(\mathbb{E}[\hat{x}(\tau, \bullet)])} \quad (104)$$

as the smoothing factor.

Proof. By the new symbol $\delta \hat{x}(\tau, \omega) := \hat{x}(\tau, \omega) - \mathbb{E}[\hat{x}(\tau, \bullet)] \in \mathbb{V}_M^4$, $\mathbb{E}[f(\hat{x}(\tau, \bullet))]$ is expanded as follows, with the introduction of a certain set $\{f_n | f_n(\mathbb{E}[\hat{x}(\tau, \bullet)]) \in \otimes_{\mathbb{V}_M^4}^n, n = 1, 2, 3, \dots\}$,

$$\begin{aligned} \mathbb{E}[f(\hat{x}(\tau, \bullet))] &= \mathbb{E} \left[f(\mathbb{E}[\hat{x}(\tau, \bullet)]) + \sum_{n=1}^{\infty} \left\langle \otimes_{\mathbb{V}_M^4}^n \delta \hat{x}(\tau, \omega), f_n(\mathbb{E}[\hat{x}(\tau, \bullet)]) \right\rangle \right] \\ &= f(\mathbb{E}[\hat{x}(\tau, \bullet)]) + \sum_{n=2}^{\infty} \left\langle \mathbb{E} \left[\otimes_{\mathbb{V}_M^4}^n \delta \hat{x}(\tau, \omega) \right], f_n(\mathbb{E}[\hat{x}(\tau, \bullet)]) \right\rangle_{\otimes_{\mathbb{V}_M^4}^n}. \end{aligned} \quad (105)$$

Where, $\langle \bullet, \bullet \rangle_{\otimes_{\mathbb{V}_M^4}^n}$ denotes the inner product $\langle \bullet, \bullet \rangle_{\otimes_{\mathbb{V}_M^4}^n} : \otimes_{\mathbb{V}_M^4}^n \times \otimes_{\mathbb{V}_M^4}^n \rightarrow \mathbb{R}$. Therefore,

$$\left\langle \mathbb{E} \left[\otimes_{\mathbb{V}_M^4}^n \delta \hat{x}(\tau, \omega) \right], f_n(\mathbb{E}[\hat{x}(\tau, \bullet)]) \right\rangle_{\otimes_{\mathbb{V}_M^4}^n}$$

represents the function with the index of $\mathbb{E}[\hat{x}(\tau, \bullet)]$. Instead of the relation (105), there is a certain C^∞ -function $\Xi : \mathbb{R} \rightarrow \mathbb{R}$ like a kind of “a gauge”,

$$\mathbb{E}[f(\hat{x}(\tau, \bullet))] = \Xi(\tau) \times f(\mathbb{E}[\hat{x}(\tau, \bullet)]), \quad (106)$$

should be satisfied. □

By using **Theorem 20**, (103) becomes

$$\mathcal{F}_{(\pm)}^{\mu\nu}(x) = -\frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \Xi(\tau') \left(\mathcal{V}_{\text{real}}^{\nu}(\mathbb{E}[\hat{x}(\tau', \bullet)]) \partial^{\mu} - \mathcal{V}_{\text{real}}^{\mu}(\mathbb{E}[\hat{x}(\tau', \bullet)]) \partial^{\nu} \right) G_{(\pm)}(x, \mathbb{E}[\hat{x}(\tau', \bullet)]). \quad (107)$$

Then, the field $\mathcal{F}_{(\pm)}^{\mu\nu}(\hat{x}(\tau, \omega))$ can be divided into the term of $\mathcal{F}_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ and the rest denoting its stochasticity;

$$\mathcal{F}_{(\pm)}^{\mu\nu}(\hat{x}(\tau, \omega)) = \mathcal{F}_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + \delta\hat{x}^{\alpha}(\tau, \omega) \cdot \partial_{\alpha} \mathcal{F}_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + O\left(\frac{2}{\otimes} \delta\hat{x}(\tau, \omega)\right). \quad (108)$$

Accompanying this expansion, the introduction of $\delta\hat{x}(\tau, \omega) := \hat{x}(\tau, \omega) - \mathbb{E}[\hat{x}(\tau, \bullet)]$ implies the following useful relations:

$$\mathcal{V}_{\text{real}}^{\alpha}(\mathbb{E}[\hat{x}(\tau, \bullet)]) - \frac{d\mathbb{E}[\hat{x}^{\alpha}(\tau, \bullet)]}{d\tau} = O\left(\frac{2}{\otimes} \delta\hat{x}(\tau, \omega)\right) \quad (109)$$

$$\frac{d\mathbb{E}[\hat{x}_{\alpha}(\tau, \bullet)]}{d\tau} \cdot \frac{d\mathbb{E}[\hat{x}^{\alpha}(\tau, \bullet)]}{d\tau} - c^2 = O\left(\frac{2}{\otimes} \delta\hat{x}(\tau, \omega)\right) \quad (110)$$

$$\frac{d\mathbb{E}[\hat{x}_{\alpha}(\tau, \bullet)]}{d\tau} \cdot \frac{d^2\mathbb{E}[\hat{x}^{\alpha}(\tau, \bullet)]}{d\tau^2} - 0 = O\left(\frac{2}{\otimes} \delta\hat{x}(\tau, \omega)\right) \quad (111)$$

Hence, the field $\mathcal{F}_{(\pm)}^{\mu\nu}(\hat{x}(\tau, \omega))$ becomes

$$\begin{aligned} \mathcal{F}_{(\pm)}^{\mu\nu}(\hat{x}(\tau, \omega)) &= -\frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \Xi(\tau') \left(\frac{d\mathbb{E}[\hat{x}^{\nu}(\tau', \bullet)]}{d\tau'} \cdot \partial^{\mu} - \frac{d\mathbb{E}[\hat{x}^{\mu}(\tau', \bullet)]}{d\tau'} \cdot \partial^{\nu} \right) \\ &\quad G_{(\pm)}(x, \mathbb{E}[\hat{x}(\tau', \bullet)]) \Big|_{x=\mathbb{E}[\hat{x}(\tau, \bullet)]} \\ &\quad - \frac{e}{c\varepsilon_0} \delta\hat{x}^{\alpha}(\tau, \omega) \int_{\mathbb{R}} d\tau' \Xi(\tau') \left(\frac{d\mathbb{E}[\hat{x}^{\nu}(\tau', \bullet)]}{d\tau'} \cdot \partial^{\mu} - \frac{d\mathbb{E}[\hat{x}^{\mu}(\tau', \bullet)]}{d\tau'} \cdot \partial^{\nu} \right) \\ &\quad \partial_{\alpha} G_{(\pm)}(x, \mathbb{E}[\hat{x}(\tau', \bullet)]) \Big|_{x=\mathbb{E}[\hat{x}(\tau, \bullet)]} + O\left(\frac{2}{\otimes} \delta\hat{x}(\tau, \omega)\right). \end{aligned} \quad (112)$$

Here ∂ denotes the partial differential for the first index of the Green function, $\partial_{\alpha} G_{(\pm)}(x, y) = \partial G_{(\pm)}(x, y) / \partial x^{\alpha}$. Then $\partial_{\alpha} G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)])$ is fulfilled the following relation via the expansion by $\delta\hat{x}(\tau, \omega)$:

$$\begin{aligned} \partial^{\mu} G_{(\pm)}(x, \mathbb{E}[\hat{x}(\tau', \bullet)]) \Big|_{x=\mathbb{E}[\hat{x}(\tau, \bullet)]} &= -\frac{\mathbb{E}[\hat{x}^{\mu}(\tau', \bullet) - \hat{x}^{\mu}(\tau, \bullet)]}{c^2 \times (\tau' - \tau)} \frac{d}{d\tau'} G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \\ &\quad \times \left[1 + \frac{(\tau' - \tau)^2}{3! \times c^2} \times \frac{d\mathbb{E}[\hat{x}^{\alpha}(\tau, \bullet)]}{d\tau} \cdot \frac{d^2\mathbb{E}[\hat{x}^{\alpha}(\tau, \bullet)]}{d\tau^2} \right. \\ &\quad \left. + O\left(\frac{2}{\otimes} \delta\hat{x}(\tau, \omega), (\tau' - \tau)^3\right) \right]. \end{aligned} \quad (113)$$

4.1 Fields along the expectation of the trajectories

The lowest term $\mathcal{F}_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ in $\mathcal{F}_{(\pm)}^{\mu\nu}(\hat{x}(\tau, \omega))$ can be concluded as follows:

Lemma 21. Consider the D -progressive $\hat{x}(\circ, \bullet)$, its expectation $\mathbb{E}[\hat{x}(\circ, \bullet)]$ and the field $\mathcal{F}_{(\pm)} \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4$ along the average trajectory $\mathbb{E}[\hat{x}(\circ, \bullet)]$. The formula of the effective radiation reaction field $\mathfrak{F}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \in$

$\mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ is modified from the Lorentz-Abraham-Dirac (LAD) scheme [30, 39], namely,

$$\begin{aligned} \mathfrak{F}(\mathbb{E}[\hat{x}(\tau, \bullet)]) &:= \frac{\mathcal{F}_{(\dot{+})}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) - \mathcal{F}_{(\dot{-})}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])}{2} \\ &= -\frac{m_0\tau_0\Xi(\tau)}{ec^2} \times \left(\begin{aligned} &\dot{a}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \otimes \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \\ &-\frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \otimes \dot{a}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \end{aligned} \right) \end{aligned} \quad (114)$$

Where $\dot{a}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \in \mathbb{V}_M^4$ denotes

$$\dot{a}(\mathbb{E}[\hat{x}(\tau, \bullet)]) := \frac{d^3\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^3} + \frac{3}{2} \frac{d \ln \Xi(\tau)}{d\tau} \frac{d^2\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^2}. \quad (115)$$

Proof. Basically, we can use the same method in [39]. $\mathcal{F}_{(\dot{\pm})}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ shall become as follows:

$$\begin{aligned} \mathcal{F}_{(\dot{\pm})}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) &= -\frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \Xi(\tau') \left(\frac{d\mathbb{E}[\hat{x}^\nu(\tau', \bullet)]}{d\tau'} \cdot \partial^\mu - \frac{d\mathbb{E}[\hat{x}^\mu(\tau', \bullet)]}{d\tau'} \cdot \partial^\nu \right) \\ &\quad \times G_{(\dot{\pm})}(x, \mathbb{E}[\hat{x}(\tau', \bullet)]) \Big|_{x=\mathbb{E}[\hat{x}(\tau, \bullet)]} \\ &= \frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \left[\begin{aligned} &\left(\Xi(\tau') \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau', \bullet)]}{d\tau'} \right) \times \mathbb{E}[\hat{x}^\mu(\tau, \bullet) - \hat{x}^\mu(\tau', \bullet)] \\ &- \left(\Xi(\tau') \cdot \frac{d\mathbb{E}[\hat{x}^\mu(\tau', \bullet)]}{d\tau'} \right) \times \mathbb{E}[\hat{x}^\nu(\tau, \bullet) - \hat{x}^\nu(\tau', \bullet)] \\ &\times \frac{\mathbb{E}[\hat{x}^\mu(\tau', \bullet) - \hat{x}^\mu(\tau, \bullet)]}{c^2 \times (\tau' - \tau)} \frac{d}{d\tau'} G_{(\dot{\pm})}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \\ &\times \left[1 + O\left((\tau' - \tau)^2, {}^2_{\otimes} \delta \hat{x}(\tau, \omega)\right) \right] \end{aligned} \right]. \end{aligned} \quad (116)$$

The part in this equation (116) should be satisfied:

$$\begin{aligned} &\left(\Xi(\tau') \cdot \frac{d}{d\tau} \mathbb{E}[\hat{x}^\nu(\tau', \bullet)] \right) \cdot \mathbb{E}[\hat{x}^\mu(\tau, \bullet) - \hat{x}^\mu(\tau', \bullet)] - \left(\Xi(\tau') \cdot \frac{d}{d\tau} \mathbb{E}[\hat{x}^\mu(\tau', \bullet)] \right) \cdot \mathbb{E}[\hat{x}^\nu(\tau, \bullet) - \hat{x}^\nu(\tau', \bullet)] \\ &= -(\tau - \tau')^2 \times \frac{\Xi(\tau)}{2} \left(\frac{d\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau} \cdot \frac{d^2\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau^2} - \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau} \cdot \frac{d^2\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau^2} \right) \\ &\quad - (\tau - \tau')^3 \times \left[\begin{aligned} &\frac{\Xi(\tau)}{3} \left(\frac{d\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau} \cdot \frac{d^3\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau^3} - \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau} \cdot \frac{d^3\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau^3} \right) \\ &+ \frac{1}{2} \frac{d\Xi(\tau)}{d\tau} \left(\frac{d\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau} \cdot \frac{d^2\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau^2} - \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau} \cdot \frac{d^2\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau^2} \right) \end{aligned} \right] \end{aligned} \quad (117)$$

The detail of the Green function should be

$$\begin{aligned} &G_{(\dot{\pm})}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \\ &= \frac{\theta(\dot{\pm})\mathbb{E}[\hat{x}^0(\tau, \bullet) - \hat{x}^0(\tau', \bullet)] \times \delta(\tau - \tau')}{4\pi c^2 \times |\tau' - \tau|} \times \left[1 + O\left((\tau' - \tau)^2, {}^2_{\otimes} \delta \hat{x}(\tau, \omega)\right) \right], \end{aligned} \quad (118)$$

where we need to consider the causality due to $\theta(\dot{\pm})\mathbb{E}[\hat{x}^0(\tau, \bullet) - \hat{x}^0(\tau', \bullet)] = \theta(\dot{\pm})(\tau - \tau')$. By substituting

this Green function for (116), the field $\mathcal{F}_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ is calculated as follows:

$$\begin{aligned} \mathcal{F}_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) &= \frac{3}{4} \frac{m_0 \tau_0 \Xi(\tau)}{ec^2} \times \left(\frac{\frac{d^2 \mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau^2} \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau}}{-\frac{d^2 \mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau^2} \cdot \frac{d\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau}} \right) \int_{\mathbb{R}} d\tau' \frac{\delta(\tau' - \tau)}{|\tau' - \tau|} \\ &\quad \mp \frac{m_0 \tau_0 \Xi(\tau)}{ec^2} \times \left(\frac{\dot{a}^\mu(\mathbb{E}[\hat{x}(\tau, \bullet)]) \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau}}{-\dot{a}^\nu(\mathbb{E}[\hat{x}(\tau, \bullet)]) \cdot \frac{d\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau}} \right) \end{aligned} \quad (119)$$

Where $\tau_0 := e^2/6\pi\epsilon_0 m_0 c^3 = O(10^{-24}\text{sec})$. For avoiding $\int_{\mathbb{R}} d\tau' \delta(\tau' - \tau)/|\tau' - \tau|$, we define the effective radiation reaction field along $\mathbb{E}[\hat{x}(\circ, \bullet)]$ like Dirac

$$\begin{aligned} \mathfrak{F}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) &:= \frac{F_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) - F_{(\mp)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])}{2} \\ &= -\frac{m_0 \tau_0 \Xi(\tau)}{ec^2} \times \left(\frac{\dot{a}^\mu(\mathbb{E}[\hat{x}(\tau, \bullet)]) \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau}}{-\dot{a}^\nu(\mathbb{E}[\hat{x}(\tau, \bullet)]) \cdot \frac{d\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau}} \right). \end{aligned} \quad (120)$$

□

4.2 Gradient of fields along the expectation of trajectories

Secondary, we consider the term denoting the stochasticity of $\mathcal{F}_{(\pm)}^{\mu\nu}(\hat{x}(\tau, \omega))$, namely, $\delta\hat{x}^\alpha(\tau, \omega) \cdot \partial_\alpha \mathcal{F}_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])$. We consider the following :

$$\begin{aligned} \partial_\alpha \mathcal{F}_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) &= -\frac{e}{c\epsilon_0} \int_{\mathbb{R}} d\tau' \Xi(\tau') \left(\frac{d\mathbb{E}[\hat{x}^\nu(\tau', \bullet)]}{d\tau'} \cdot \partial^\mu - \frac{d\mathbb{E}[\hat{x}^\mu(\tau', \bullet)]}{d\tau'} \cdot \partial^\nu \right) \\ &\quad \partial_\alpha G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \end{aligned} \quad (121)$$

The factor $\partial^\mu \partial_\alpha G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)])$ in this equation becomes

$$\begin{aligned} &\partial^\mu \partial_\alpha G_{(\pm)}(x, \mathbb{E}[\hat{x}(\tau', \bullet)]) \Big|_{x=\mathbb{E}[\hat{x}(\tau, \bullet)]} \\ &= \frac{\delta_\alpha^\mu}{c^2(\tau' - \tau)} \times \frac{d}{d\tau'} G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \\ &\quad \times \left[1 + \frac{(\tau' - \tau)^2}{3!} \times T^{-2}(\tau) + O((\tau' - \tau)^3) \right] \\ &\quad + \frac{\mathbb{E}[\hat{x}_\alpha(\tau, \bullet) - \hat{x}_\alpha(\tau', \bullet)] \cdot \frac{d\mathbb{E}[\hat{x}^\mu(\tau', \bullet)]}{d\tau'}}{c^4(\tau' - \tau)^2} \times \frac{d}{d\tau'} G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \\ &\quad \times \left[1 + \frac{(\tau' - \tau)^2}{3} \times T^{-2}(\tau) + O((\tau' - \tau)^3) \right] \\ &\quad + \frac{\mathbb{E}[\hat{x}^\mu(\tau, \bullet) - \hat{x}^\mu(\tau', \bullet)] \cdot \mathbb{E}[\hat{x}_\alpha(\tau, \bullet) - \hat{x}_\alpha(\tau', \bullet)]}{c^4(\tau' - \tau)^2} \end{aligned}$$

$$\begin{aligned}
& \times \frac{d^2}{d\tau'^2} G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \\
& \times \left[1 + \frac{(\tau' - \tau)^2}{3} \times T^{-2}(\tau) + O((\tau' - \tau)^3) \right].
\end{aligned} \tag{122}$$

Where, T^{-2} denotes

$$T^{-2}(\tau) := \frac{1}{c^2} \times \frac{d^2 \mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau^2} \cdot \frac{d^2 \mathbb{E}[\hat{x}^\alpha(\tau, \bullet)]}{d\tau^2}. \tag{123}$$

Hence,

$$\begin{aligned}
& - \left[\left(\Xi(\tau') \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau', \bullet)]}{d\tau} \right) \cdot \partial^\mu - (\mu \leftrightarrow \nu) \right] \partial_\alpha G_{(\pm)}(x, \mathbb{E}[\hat{x}(\tau', \bullet)]) \Big|_{x=\mathbb{E}[\hat{x}(\tau, \bullet)]} \\
& = - \frac{\left(\Xi(\tau') \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau', \bullet)]}{d\tau'} \right) \cdot \delta_\alpha^\mu - (\mu \leftrightarrow \nu)}{c^2(\tau' - \tau)} \times \frac{d}{d\tau'} G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \\
& \quad \times \left[1 + \frac{(\tau' - \tau)^2}{6} \times T^{-2}(\tau) + O((\tau' - \tau)^3) \right] \\
& \quad - \frac{\left(\Xi(\tau') \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau', \bullet)]}{d\tau} \right) \cdot \mathbb{E}[\hat{x}^\mu(\tau, \bullet) - \hat{x}^\mu(\tau', \bullet)] - (\mu \leftrightarrow \nu)}{c^4(\tau' - \tau)^2} \\
& \quad \times \mathbb{E}[\hat{x}_\alpha(\tau, \bullet) - \hat{x}_\alpha(\tau', \bullet)] \cdot \frac{d^2}{d\tau'^2} G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \\
& \quad \times \left[1 + \frac{(\tau' - \tau)^2}{3} \times T^{-2}(\tau) + O((\tau' - \tau)^3) \right].
\end{aligned} \tag{124}$$

Therefore, $\partial_\alpha \mathcal{F}_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ is transformed as follows:

$$\begin{aligned}
& \partial_\alpha \mathcal{F}_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \\
& = - \frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \frac{\left(\Xi(\tau') \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau', \bullet)]}{d\tau} \right) \cdot \delta_\alpha^\mu - (\mu \leftrightarrow \nu)}{c^2(\tau' - \tau)} \\
& \quad \times \frac{d}{d\tau'} G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \\
& \quad \times \left[1 + \frac{(\tau' - \tau)^2}{6} \times T^{-2}(\tau) + O((\tau' - \tau)^3) \right] \\
& \quad - \frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \frac{\mathbb{E}[\hat{x}_\alpha(\tau, \bullet) - \hat{x}_\alpha(\tau', \bullet)]}{c^4(\tau' - \tau)^2} \\
& \quad \times \left[\left(\Xi(\tau') \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau', \bullet)]}{d\tau} \right) \cdot \mathbb{E}[\hat{x}^\mu(\tau, \bullet) - \hat{x}^\mu(\tau', \bullet)] - (\mu \leftrightarrow \nu) \right] \\
& \quad \times \frac{d^2}{d\tau'^2} G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \\
& \quad \times \left[1 + \frac{(\tau' - \tau)^2}{3} \times T^{-2}(\tau) + O((\tau' - \tau)^3) \right]
\end{aligned} \tag{125}$$

The first component in this equation is

$$\begin{aligned}
& -\frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \frac{\left(\Xi(\tau') \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau', \bullet)]}{d\tau} \right)}{c^2(\tau' - \tau)} \cdot \delta_\alpha^\mu - (\mu \leftrightarrow \nu) \\
& \quad \times \frac{d}{d\tau'} G_{(\pm)}(\mathbb{E}[\hat{x}_\beta(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \times \left[1 + \frac{(\tau' - \tau)^2}{6} \times T^{-2}(\tau) + O((\tau' - \tau)^3) \right] \\
& = \frac{m_0\tau_0}{ec^2} \int_{\mathbb{R}} d\tau' \frac{\delta(\tau' - \tau)}{|\tau' - \tau|} \times \left[\begin{aligned} & -\frac{3}{2} \times \frac{\delta_\alpha^\mu \cdot \left(\Xi(\tau) \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau} \right)}{|\tau' - \tau|^2} \\ & + \frac{3}{4} \times \delta_\alpha^\mu \cdot \frac{d^2}{d\tau^2} \left(\Xi(\tau) \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau} \right) \end{aligned} \right] \\
& \quad - (\mu \leftrightarrow \nu) \\
& \quad \mp \frac{m_0\tau_0}{ec^2} \left[\begin{aligned} & \frac{1}{2} \times \delta_\alpha^\mu \cdot \frac{d^3}{d\tau^3} \left(\Xi(\tau) \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau} \right) \\ & + \frac{T^{-2}(\tau)}{2} \times \delta_\alpha^\mu \cdot \frac{d}{d\tau} \left(\Xi(\tau) \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau} \right) \end{aligned} \right] - (\mu \leftrightarrow \nu). \tag{126}
\end{aligned}$$

The second term becomes,

$$\begin{aligned}
& -\frac{e}{c\varepsilon_0} \int_{\mathbb{R}} d\tau' \frac{\mathbb{E}[\hat{x}_\alpha(\tau, \bullet) - \hat{x}_\alpha(\tau', \bullet)]}{c^4(\tau' - \tau)^2} \\
& \quad \times \left[\left(\Xi(\tau) \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau} \right) \cdot \mathbb{E}[\hat{x}^\mu(\tau, \bullet) - \hat{x}^\mu(\tau', \bullet)] - (\mu \leftrightarrow \nu) \right] \\
& \quad \times \frac{d^2}{d\tau'^2} G_{(\pm)}(\mathbb{E}[\hat{x}(\tau, \bullet)], \mathbb{E}[\hat{x}(\tau', \bullet)]) \times \left[1 + \frac{(\tau' - \tau)^2}{3} \times T^{-2}(\tau) + O((\tau' - \tau)^3) \right] \\
& = -\frac{m_0\tau_0}{ec^4} \int_{\mathbb{R}} d\tau' \frac{\delta(\tau' - \tau)}{|\tau' - \tau|} \left[\begin{aligned} & \left(\frac{3}{2} \frac{d\Xi(\tau)}{d\tau} \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} + \frac{3}{4} \Xi(\tau) \frac{d^2\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau^2} \right) A^{\mu\nu}(\tau) \\ & + \frac{1}{2} \Xi(\tau) \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} B^{\mu\nu}(\tau) \end{aligned} \right] \\
& \quad \pm \frac{m_0\tau_0}{ec^4} \left[\begin{aligned} & \left(\frac{9}{4} \frac{d^2\Xi(\tau)}{d\tau^2} \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} + \frac{9}{4} \frac{d\Xi(\tau)}{d\tau} \frac{d^2\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau^2} \right) A^{\mu\nu}(\tau) \\ & + \frac{3}{4} \Xi(\tau) \frac{d^3\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau^3} - \frac{3}{2} T^{-2}(\tau) \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} \\ & + \left(3 \frac{d\Xi(\tau)}{d\tau} \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} + \frac{3}{2} \Xi(\tau) \frac{d^2\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau^2} \right) B^{\mu\nu}(\tau) \\ & + \frac{9}{8} \Xi(\tau) \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} C^{\mu\nu}(\tau) + \frac{3}{4} \Xi(\tau) \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} D^{\mu\nu}(\tau) \end{aligned} \right]. \tag{127}
\end{aligned}$$

Here the following symbols are employed:

$$A(\tau) := \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \otimes \frac{d^2\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^2} - \frac{d^2\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^2} \otimes \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \tag{128}$$

$$B(\tau) := \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \otimes \frac{d^3\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^3} - \frac{d^3\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^3} \otimes \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \tag{129}$$

$$C(\tau) := \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \otimes \frac{d^4\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^4} - \frac{d^4\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^4} \otimes \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau}. \quad (130)$$

$$D(\tau) := \frac{d^2\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^2} \otimes \frac{d^3\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^3} - \frac{d^3\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^3} \otimes \frac{d^2\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau^2} \quad (131)$$

Lemma 22. Consider the D -progressive $\hat{x}(\circ, \bullet)$, its expectation $\mathbb{E}[\hat{x}(\circ, \bullet)]$ and the field $\mathcal{F}_{(\pm)}^{\mu\nu} \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ along the average trajectory $\mathbb{E}[\hat{x}(\circ, \bullet)]$. The formula of $\partial_\alpha \mathfrak{F}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ which is the gradient of the field avoiding the singularity is as follows:

$$\begin{aligned} & \partial_\alpha \mathfrak{F}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \\ &:= \frac{\partial_\alpha \mathcal{F}_{(\pm)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) - \partial_\alpha \mathcal{F}_{(\mp)}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])}{2} \\ &= -\frac{m_0\tau_0}{ec^2} \left[\begin{aligned} & \frac{1}{2} \times \delta_\alpha^\mu \cdot \frac{d^3}{d\tau^3} \left(\Xi(\tau) \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau} \right) \\ & + \frac{T^{-2}(\tau)}{2} \times \delta_\alpha^\mu \cdot \frac{d}{d\tau} \left(\Xi(\tau) \cdot \frac{d\mathbb{E}[\hat{x}^\nu(\tau, \bullet)]}{d\tau} \right) \\ & - (\mu \leftrightarrow \nu) \end{aligned} \right] \\ &+ \frac{m_0\tau_0}{ec^4} \left[\begin{aligned} & \left(\frac{9}{4} \frac{d^2\Xi(\tau)}{d\tau^2} \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} + \frac{9}{4} \frac{d\Xi(\tau)}{d\tau} \frac{d^2\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau^2} \right) A^{\mu\nu}(\tau) \\ & + \left(\frac{3}{4} \Xi(\tau) \frac{d^3\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau^3} - \frac{3}{2} T^{-2}(\tau) \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} \right) B^{\mu\nu}(\tau) \\ & + \left(3 \frac{d\Xi(\tau)}{d\tau} \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} + \frac{3}{2} \Xi(\tau) \frac{d^2\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau^2} \right) C^{\mu\nu}(\tau) \\ & + \frac{9}{8} \Xi(\tau) \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} D^{\mu\nu}(\tau) \end{aligned} \right] \quad (132) \end{aligned}$$

4.3 Effective radiation reaction field

Hereby, let us conclude the topics of radiation reaction: At first, consider the problem of the singularity of the radiation field. Now we find the two solutions of the Maxwell equation, namely, the retarded field $\mathcal{F}_{(\pm)}(x) \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ and the advanced field $\mathcal{F}_{(\mp)}(x) \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$. The definitions of fields were as follows:

$$\mathcal{F}_{(\pm)}^{\mu\nu}(x) = F_{(\pm)}^{\mu\nu}(x) + \delta f_{(\pm)}^{\mu\nu}(x) \quad (133)$$

$$F_{(\pm)}^{\mu\nu}(x) := \partial^\mu A_{(\pm)}^\nu(x) - \partial^\nu A_{(\pm)}^\mu(x) \quad (134)$$

$$\delta f_{(\pm)}^{\mu\nu}(x) := \partial^\mu \delta a_{(\pm)}^\nu(x) - \partial^\nu \delta a_{(\pm)}^\mu(x) \quad (135)$$

And we found the fact $\mathcal{F}_{(\pm)}^{\mu\nu}(x)$ includes the term of the singularity. We regard the term $\delta f_{(\pm)}(x) \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ as the absorber of the singularity such that

$$\partial_\mu \left[\mathcal{F}_{(\pm)}^{\mu\nu}(x) - \delta f_{(\pm)}^{\mu\nu}(x) \right] = 0. \quad (136)$$

Then, the field $F^{\mu\nu}(x) := \pm F_{(\pm)}^{\mu\nu}(x)$ describes the homogeneous field; $\partial_\mu F^{\mu\nu}(x) = 0$. Combining the relation of them,

$$\boxed{\mathcal{F}_{(\pm)}(\hat{x}(\tau, \omega)) = \pm F(\hat{x}(\tau, \omega)) + \delta f(\hat{x}(\tau, \omega))}. \quad (137)$$

When we define the initial condition at $\tau = -\infty$, the solution of the Maxwell equation shall be the retarded field $\mathcal{F}_{(\pm)} = \pm F + \delta f$. Therefore, the set of the dynamics is as follows:

$$m_0 \mathfrak{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) = -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F^{\mu\nu}(\hat{x}(\tau, \omega)) \quad (138)$$

$$\partial_\mu [F^{\mu\nu}(x) + \delta f^{\mu\nu}(x)] = \mu_0 \times \mathbb{E} \left[-ec \int_{\mathbb{R}} d\tau \operatorname{Re} \{ \mathcal{V}^\nu(x) \} \delta^4(x - \hat{x}(\tau, \bullet)) \right] \quad (139)$$

Then, $F = F_{\text{ex}} + \mathfrak{F}$ such that $\partial_\mu F^{\mu\nu} = 0$ in general.

Theorem 23 (Radiation reaction). *Consider the D-progressive $\hat{x}(\circ, \bullet)$ as a spin-less electron's motion which draws its trajectory in the Minkowski spacetime $(\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_{\text{M}}^4, g)), \mu)$. Here, the following Maxwell equation is the mechanism of the field-generation:*

$$\partial_\mu \mathcal{F}_{(\pm)}^{\mu\nu}(x) = \mu_0 \times \mathbb{E} \left[-ec \int_{\mathbb{R}} d\tau \operatorname{Re} \{ \mathcal{V}^\nu(x) \} \delta^4(x - \hat{x}(\tau, \bullet)) \right] \quad (140)$$

For the retarded field $\mathcal{F}_{(\pm)} \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4$ and the advanced fields $\mathcal{F}_{(\pm)} \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4$. When we select the causality from the past to the present, the solution is fixed only the retarded field; $\mathcal{F}_{(\pm)} = \pm F + \delta f$. The field $F \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4$ such that $\partial_\mu F^{\mu\nu} = 0$ becomes

$$F := \frac{\mathcal{F}_{(\pm)}(\hat{x}(\tau, \omega)) - \mathcal{F}_{(\pm)}(\hat{x}(\tau, \omega))}{2} \quad (141)$$

and the singularity is

$$\delta f := \frac{\mathcal{F}_{(\pm)}(\hat{x}(\tau, \omega)) + \mathcal{F}_{(\pm)}(\hat{x}(\tau, \omega))}{2} \quad (142)$$

at $x = \hat{x}(\tau, \omega)$. The homogeneous field F is separated to the two elementary solutions, an arbitrary external field part $F_{\text{ex}} \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4$ and the radiation reaction field part $\mathfrak{F} \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4$. The radiation reaction field \mathfrak{F} stabilized its singularity at the point $x = \hat{x}(\tau, \omega)$, acts on its spin-less electron:

$$\mathfrak{F}(\hat{x}(\tau, \omega)) = \mathfrak{F}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + \delta \hat{x}^\alpha(\tau, \omega) \cdot \partial_\alpha \mathfrak{F}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + O\left(\frac{2}{\otimes} \delta \hat{x}(\tau, \omega)\right) \in \mathbb{V}_{\text{M}}^4 \otimes \mathbb{V}_{\text{M}}^4 \quad (143)$$

The details of the each terms are expressed by **Lemma 21** and **Lemma 22**. Since it is the homogeneous solution, i.e., $\partial_\mu \mathfrak{F}^{\mu\nu} = 0$ should be fulfilled. As the general scattering in the class of a single spin-less electron (see **Definition 1**), the dynamics of this radiating spin-less electron is described by

$$\boxed{m_0 \mathfrak{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) = -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) [F_{\text{ex}}^{\mu\nu}(\hat{x}(\tau, \omega)) + \mathfrak{F}^{\mu\nu}(\hat{x}(\tau, \omega))]} \quad (144)$$

The dynamics of the Klein-Gordon particle drawing a D-progressively measurable process with the radiation reaction effect is hereby derived. By tracking the motion of a scalar electron by this equation of motion, we can find the general scatterings between this scalar electron and photonic fields.

5 Conclusion and discussion

We discussed the formulation of the kinematics and the dynamics of a stochastic spin-less electron equivalent to the Klein-Gordon particle interacting with fields for the purpose of the new description of radiation reaction in high-intensity field physics. For realizing this expression, we considered the kinematics of a relativistic-Brownian particle as the D-progressive $\hat{x}(\circ, \bullet)$ drawing its trajectory in the Minkowski spacetime $(\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu)$ at **Definition 4** in Ch.2. Here, we needed to consider the probability density $p : \mathbb{A}^4(\mathbb{V}_M^4, g) \times \mathbb{R} \rightarrow [0, \infty)$ and the definition of the proper time $d\tau$ (51) as the mimic of classical dynamics. The complex differential \hat{d} (29) and the complex velocity $\hat{\mathcal{V}}$ (33) which are the main casts of the present model were also introduced for its writing. In Ch.3, the dynamics of the stochastic particle was proposed. We introduced the new action integral (73-74) corresponding to the form in classical dynamics. Hence, we could obtain the dynamics of a stochastic particle and fields via the equations of their motions by the variations of this action integral. The special remarks at here are (A) this method can derive the Maxwell equation coupled with the current of a stochastic particle (see (96)) and (B) the dynamics of the stochastic particle (81-82) induces Ehrenfest's theorem of the Klein-Gordon particle as its average behavior. Then by using these ideas, the equation of a radiating spin-less electron (144) was derived in Ch.4. Let us summarize the results of this article.

Conclusion 24 (System of a radiating spin-less electron). Consider the probability space $(\Omega, D(\mathcal{P}), \mathcal{P})$. When the sub- σ -algebras of $\mathcal{P}_{\tau \in \mathbb{R}}$ and $\mathcal{F}_{\tau \in \mathbb{R}}$ with filtration are included in $D(\mathcal{P})$, the D-progressive $\hat{x}(\circ, \bullet) := \{\hat{x}(\tau, \omega) \in \mathbb{A}^4(\mathbb{V}_M^4, g) | \tau \in \mathbb{R}, \omega \in \Omega\}$ can be defined as the trajectory of a stochastic spin-less electron in the Minkowski spacetime $(\mathbb{A}^4(\mathbb{V}_M^4, g), \mathcal{B}(\mathbb{A}^4(\mathbb{V}_M^4, g)), \mu)$. The action integral of (73-74) provides the following dynamics of a stochastic spin-less electron and a field characterized by $\mathcal{V} \in \mathbb{V}_M^4 \oplus i\mathbb{V}_M^4$ and $F \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$:

$$m_0 \mathfrak{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) = -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) [F_{\text{ex}}^{\mu\nu}(\hat{x}(\tau, \omega)) + \mathfrak{F}^{\mu\nu}(\hat{x}(\tau, \omega))] \quad (145)$$

$$\partial_\mu [F_{\text{ex}}^{\mu\nu}(x) + \mathfrak{F}^{\mu\nu}(x) + \delta f^{\mu\nu}(x)] = \mu_0 \times \mathbb{E}_\omega \left[-ec \int_{\mathbb{R}} d\tau' \text{Re} \{ \mathcal{V}^\nu(x) \} \delta^4(x - \hat{x}(\tau', \omega)) \right] \quad (146)$$

Here, the dynamics of (145) is equivalent to the Klein-Gordon equation. $\delta f \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ is the term of the singularities in the retarded field $\mathcal{F}_{(+)} \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ and the advanced field $\mathcal{F}_{(-)} \in \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$. Therefore, $F_{\text{ex}} + \mathfrak{F} = [\mathcal{F}_{(+)} - \mathcal{F}_{(-)}]/2$ represents the general homogeneous solution of (146). Since F_{ex} is the external field, radiation reaction is dominated by the field \mathfrak{F} :

$$\begin{aligned} \mathfrak{F}(\hat{x}(\tau, \omega)) &= \mathfrak{F}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + \delta \hat{x}^\alpha(\tau, \omega) \cdot \partial_\alpha \mathfrak{F}(\mathbb{E}[\hat{x}(\tau, \bullet)]) + O\left(\frac{2}{\otimes} \delta \hat{x}(\tau, \omega)\right) \\ &= -\frac{m_0 \tau_0 \Xi(\tau)}{ec^2} \left(\dot{a}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \otimes \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \right. \\ &\quad \left. - \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \otimes \dot{a}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \right) \\ &\quad - \frac{1}{2} \frac{m_0 \tau_0}{ec^2} \left[\delta \hat{x}(\tau, \omega) \otimes \left(\frac{d^2}{d\tau^2} + T^{-2}(\tau) \frac{d}{d\tau} \right) \left(\Xi(\tau) \cdot \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \right) \right. \\ &\quad \left. - \left(\frac{d^2}{d\tau^2} + T^{-2}(\tau) \frac{d}{d\tau} \right) \left(\Xi(\tau) \cdot \frac{d\mathbb{E}[\hat{x}(\tau, \bullet)]}{d\tau} \right) \otimes \delta \hat{x}(\tau, \omega) \right] \end{aligned}$$

$$+ \frac{3}{8} \frac{m_0 \tau_0}{e c^4} \delta \hat{x}^\alpha(\tau, \omega) \cdot \left\{ \begin{aligned} & \left(6 \frac{d^2 \Xi(\tau)}{d\tau^2} + 6 \frac{d\Xi(\tau)}{d\tau} \frac{d}{d\tau} \right) \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} \times A(\tau) \\ & + 2\Xi(\tau) \frac{d^2}{d\tau^2} - 4T^{-2}\Xi(\tau) \left(8 \frac{d\Xi(\tau)}{d\tau} + 4\Xi(\tau) \frac{d}{d\tau} \right) \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} \times B(\tau) \\ & + \Xi(\tau) \frac{d\mathbb{E}[\hat{x}_\alpha(\tau, \bullet)]}{d\tau} \times [3C(\tau) + 2D(\tau)] \end{aligned} \right\}. \quad (147)$$

Hence, the full dynamics of the radiating spin-less electron is as follows:

$$\boxed{m_0 \mathfrak{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) = -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F_{\text{ex}}^{\mu\nu}(\hat{x}(\tau, \omega)) - e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) \left[\begin{aligned} & \mathfrak{F}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \\ & + \delta \hat{x}^\alpha(\tau, \omega) \cdot \partial_\alpha \mathfrak{F}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \\ & + O\left(\frac{2}{\otimes} \delta \hat{x}(\tau, \omega)\right) \end{aligned} \right]} \quad (148)$$

This is the quantized equation from the LAD equation in classical dynamics,

$$m_0 \frac{dv^\mu}{d\tau} = -e v_\nu F_{\text{ex}}^{\mu\nu} + \frac{m_0 \tau_0}{c^2} \left(\frac{d^2 v^\mu}{d\tau^2} v^\nu - \frac{d^2 v^\nu}{d\tau^2} v^\mu \right) v_\nu. \quad (149)$$

Though the dynamics of a radiating stochastic particle corresponding to the LAD equation was derived, however (148) includes many higher order derivatives. The LAD equation has the exponential factor $dw^\mu/d\tau \propto \exp(\tau/\tau_0)$, namely, there is the run-away problem as its mathematical difficulties. In the realistic applications of this model, it may suffer us in its estimations and numerical simulations in its experimental designs. Let us also introduce the term reduction like the Ford-O'Connell [42]/Landau-Lifshitz [43] schemes. In the case of by Landau-Lifshitz, it is carried out by the following perturbation with respect to $\tau_0 = O(10^{-24}\text{sec})$,

$$\frac{dv^\mu}{d\tau} = -\frac{e}{m_0} F_{\text{ex}}^{\mu\nu} v_\nu + O(\tau_0), \quad (150)$$

$$\frac{d^2 v^\mu}{d\tau^2} = -\frac{e}{m_0} \frac{dF_{\text{ex}}^{\mu\nu}}{d\tau} v_\nu + \frac{e^2}{m_0^2} g_{\alpha\beta} F_{\text{ex}}^{\mu\alpha} F_{\text{ex}}^{\beta\nu} v_\nu + O(\tau_0), \quad (151)$$

for the term of $m_0 \tau_0 / c^2 \times (d^2 v^\mu / d\tau^2 \cdot v^\nu - d^2 v^\nu / d\tau^2 \cdot v^\mu) v_\nu$ in the LAD equation (149). In our present case of the stochastic model, the interaction with the external field is expressed by

$$f_{\text{ex}}^\mu(\hat{x}(\tau, \omega)) := -e \hat{\mathcal{V}}_\nu(\hat{x}(\tau, \omega)) F_{\text{ex}}^{\mu\nu}(\hat{x}(\tau, \omega)) \in \mathbb{V}_{\text{M}}^4 \oplus i\mathbb{V}_{\text{M}}^4 \quad (152)$$

and consider Ehrenfest's theorem of (148):

$$\frac{d^2 \mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau^2} = \frac{1}{m_0} \times \text{Re} \left\{ \frac{d}{d\tau} \mathbb{E}[\mathcal{V}^\mu(\hat{x}(\tau, \bullet))] \right\}$$

$$= -\frac{e}{m_0} F_{\text{ex}}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \frac{d\mathbb{E}[\hat{x}_\nu(\tau, \bullet)]}{d\tau} + O\left(\tau_0, {}^2\delta\hat{x}(\tau, \omega)\right) \quad (153)$$

$$\begin{aligned} \frac{d^3\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau^3} &= -\frac{e}{m_0} \frac{dF_{\text{ex}}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])}{d\tau} \frac{d\mathbb{E}[\hat{x}_\nu(\tau, \bullet)]}{d\tau} \\ &\quad + \frac{e^2}{m_0^2} g_{\alpha\beta} F_{\text{ex}}^{\mu\alpha}(\mathbb{E}[\hat{x}(\tau, \bullet)]) F_{\text{ex}}^{\beta\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \frac{d\mathbb{E}[\hat{x}_\nu(\tau, \bullet)]}{d\tau} \\ &\quad + O\left(\tau_0, {}^2\delta\hat{x}(\tau, \omega)\right) \end{aligned} \quad (154)$$

Where the following relation is employed,

$$\begin{aligned} f_{\text{Re}}^\mu(\hat{x}(\tau, \omega)) &:= \text{Re}\{f_{\text{ex}}^\mu(\hat{x}(\tau, \omega))\} \\ &= -eF_{\text{ex}}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \frac{d\mathbb{E}[\hat{x}_\nu(\tau, \bullet)]}{d\tau} + O\left(\tau_0, {}^2\delta\hat{x}(\tau, \omega)\right). \end{aligned} \quad (155)$$

In order to the above discussion, the classical like equations are also satisfied:

$$\frac{d\mathbb{E}[\hat{x}_\mu(\tau, \bullet)]}{d\tau} \cdot \frac{d^2\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau^2} = 0 + O\left(\tau_0^2, {}^2\delta\hat{x}(\tau, \omega)\right) \quad (156)$$

$$\frac{d\mathbb{E}[\hat{x}_\mu(\tau, \bullet)]}{d\tau} \cdot f_{\text{Re}}^\mu(\hat{x}(\tau, \omega)) = 0 + O\left(\tau_0^2, {}^2\delta\hat{x}(\tau, \omega)\right) \quad (157)$$

By this expansion,

$$\begin{aligned} \dot{a}^\mu(\mathbb{E}[\hat{x}(\tau, \bullet)]) &= \frac{3}{2} \frac{d \ln \Xi(\tau)}{d\tau} \frac{f_{\text{Re}}^\mu(\mathbb{E}[\hat{x}(\tau, \bullet)])}{m_0} - \frac{e}{m_0} \frac{dF_{\text{ex}}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])}{d\tau} \frac{d\mathbb{E}[\hat{x}_\nu(\tau, \bullet)]}{d\tau} \\ &\quad - \frac{e}{m_0^2} g_{\alpha\beta} F_{\text{ex}}^{\mu\alpha}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \cdot f_{\text{Re}}^\beta(\mathbb{E}[\hat{x}(\tau, \bullet)]) + O\left(\tau_0, {}^2\delta\hat{x}(\tau, \omega)\right) \end{aligned} \quad (158)$$

is induced. Hence, the following scheme shall be realized:

Lemma 25 (Approximation). *Consider the dynamics of a D -progressive radiating spin-less electron, (148). This equation can be perturbed with respect to the parameter of $\tau_0 = O(10^{-24}\text{sec})$ as follows:*

$$\begin{aligned} &m_0 \mathfrak{D}_\tau \mathcal{V}^\mu(\hat{x}(\tau, \omega)) \\ &= f_{\text{ex}}^\mu(\hat{x}(\tau, \omega)) + \frac{3}{2} \tau_0 \frac{d\Xi(\tau)}{d\tau} \times f_{\text{Re}}^\mu(\mathbb{E}[\hat{x}(\tau, \bullet)]) \\ &\quad - e\tau_0 \Xi(\tau) \frac{dF_{\text{ex}}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])}{d\tau} \frac{d\mathbb{E}[\hat{x}_\nu(\tau, \bullet)]}{d\tau} \\ &\quad - \frac{e\tau_0 \Xi(\tau)}{m_0} g_{\alpha\beta} F_{\text{ex}}^{\mu\alpha}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \cdot f_{\text{Re}}^\beta(\mathbb{E}[\hat{x}(\tau, \bullet)]) \\ &\quad + \frac{\tau_0 \Xi(\tau)}{m_0 c^2} \langle f_{\text{Re}}(\mathbb{E}[\hat{x}(\tau, \bullet)]), f_{\text{Re}}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \rangle_{\mathbb{V}_M^4} \frac{d\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau} \\ &\quad + \delta\hat{x}^\alpha(\tau, \omega) \cdot \mathfrak{A}_\alpha^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \frac{d\mathbb{E}[\hat{x}_\nu(\tau, \bullet)]}{d\tau} + O\left(\tau_0^2, {}^2\delta\hat{x}(\tau, \omega)\right) \end{aligned} \quad (159)$$

Where, $\langle \bullet, \bullet \rangle_{\mathbb{V}_M^4} : \mathbb{V}_M^4 \times \mathbb{V}_M^4 \rightarrow \mathbb{R}$ is the inner product of vectors, $\mathfrak{A}_\alpha^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \in (\mathbb{V}_M^4)^* \otimes \mathbb{V}_M^4 \otimes \mathbb{V}_M^4$ is the perturbed tensor of $\partial_\alpha \mathfrak{F}^{\mu\nu}(\mathbb{E}[\hat{x}(\tau, \bullet)])$ and $f_{\text{Re}}(\mathbb{E}[\hat{x}(\tau, \bullet)]) := \text{Re}\{f_{\text{ex}}(\mathbb{E}[\hat{x}(\tau, \bullet)])\}$. This is the stochastic version of Landau-Lifshitz's approximation.

The most attractive result in high-intensity field physics is the average of this equation of motion, (159) as its classical behavior.

$$\begin{aligned}
& m_0 \frac{d^2}{d\tau^2} \mathbb{E}[\hat{x}^\mu(\tau, \bullet)] \\
&= \left(1 + \frac{3}{2} \tau_0 \frac{d\Xi(\tau)}{d\tau} \right) \times f_{\text{Re}}^\mu(\mathbb{E}[\hat{x}(\tau, \bullet)]) \\
&\quad - e\tau_0 \Xi(\tau) g_{\alpha\beta} \frac{dF_{\text{ex}}^{\mu\alpha}(\mathbb{E}[\hat{x}(\tau, \bullet)])}{d\tau} \cdot \frac{d\mathbb{E}[\hat{x}^\beta(\tau, \bullet)]}{d\tau} \\
&\quad - \frac{e\tau_0 \Xi(\tau)}{m_0} g_{\alpha\beta} F_{\text{ex}}^{\mu\alpha}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \cdot f_{\text{Re}}^\beta(\mathbb{E}[\hat{x}(\tau, \bullet)]) \\
&\quad + \frac{\tau_0 \Xi(\tau)}{m_0 c^2} \langle f_{\text{Re}}(\mathbb{E}[\hat{x}(\tau, \bullet)]), f_{\text{Re}}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \rangle_{\mathbb{V}_M^4} \frac{d\mathbb{E}[\hat{x}^\mu(\tau, \bullet)]}{d\tau} + O\left(\frac{2}{\tau_0^2} \delta \hat{x}(\tau, \omega)\right) \quad (160)
\end{aligned}$$

This quasi Landau-Lifshitz equation is characterized by the smoothing factor $\Xi(\tau)$. When $\Xi \equiv 1$, it converges completely to the Landau-Lifshitz equation in classical dynamics [43]. We can find the radiation formula in the final term of the RHS, namely, Larmor's radiation formula in “the direct radiation term”:

$$\frac{dW}{dt} = -\frac{\tau_0 \Xi(\tau)}{m_0} \langle f_{\text{Re}}(\mathbb{E}[\hat{x}(\tau, \bullet)]), f_{\text{Re}}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \rangle_{\mathbb{V}_M^4} + O\left(\tau_0^2, \frac{2}{\tau_0^2} \delta \hat{x}(\tau, \omega)\right) \quad (161)$$

The smoothing factor $\Xi(\tau)$ should be derived from

$$\mathbb{E}[f(\hat{x}(\tau, \bullet))] = \Xi(\tau) \times f(\mathbb{E}[\hat{x}(\tau, \bullet)]) \quad (162)$$

with the Fokker-Planck equation for the probability density function (38-39). However, when we can assume the non-linear Compton scattering (the single photon emission process) by an external “plane wave” field (**Example 2**), the synchrotron radiation formula from scalar QED [14] can be applicable. In this case, it is known

$$\Xi(\tau) = \frac{9\sqrt{3}}{8\pi} \int_0^{\chi^{-1}} dr r \int_{\frac{r}{1-\chi r}}^\infty dr' K_{5/3}(r') \quad (163)$$

with the definition of the non-linearity parameter,

$$\chi := \frac{3}{2} \frac{\hbar}{m_0^2 c^3} \sqrt{-\langle f_{\text{Re}}(\mathbb{E}[\hat{x}(\tau, \bullet)]), f_{\text{Re}}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \rangle_{\mathbb{V}_M^4}}. \quad (164)$$

Since r is characterized by the relation $r := \chi^{-1} \times \hbar\omega/m_0 c^2 \gamma$, the radiation spectrum can be introduced as follows:

$$\frac{d^2 W}{dtd(\hbar\omega)} = -\frac{\tau_0 \Xi(\tau)}{m_0} \langle f_{\text{Re}}(\mathbb{E}[\hat{x}(\tau, \bullet)]), f_{\text{Re}}(\mathbb{E}[\hat{x}(\tau, \bullet)]) \rangle_{\mathbb{V}_M^4} \times \frac{d\Xi(\tau)}{d(\hbar\omega)} \quad (165)$$

$$\frac{d\Xi(\tau)}{d(\hbar\omega)} = \frac{9\sqrt{3}}{8\pi\chi^2} \frac{\hbar\omega}{(m_0 c^2 \gamma)^2} \int_{\frac{r}{1-\chi r}}^\infty dr' K_{5/3}(r') \quad (166)$$

In the fact, the averaged trajectory via the substitution (163) for (160) convargers to the previous results [16, 17]. Hence, the biggest key for high-intensity field physics is **Theorem 20**, the appearance of the smoothing factor $\Xi : \mathbb{R} \rightarrow \mathbb{R}$ being able to represent the non-linearity of quantum dynamics. The critical problem in the radiation reaction experiment is how to detect this smoothing factor Ξ .

As the further works, the investigations of the deeper analysis of this method and numerical simulations have to be expected to innovate high-intensity field physics from radiation reaction toward together with real experiments carried out by the state-of-the-arts $O(10\text{PW})$ lasers.

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